Circular Choosability

Frédéric Havet, Ross Kang, Tobias Müller, Jean-Sébastien Sereni

INSTITUT NATIONAL DE RECHERCHE EN INFORMATIQUE ET EN AUTOMATIQUE

Circular choosability

Frédéric Havet — Ross J. Kang — Tobias Müller — Jean-Sébastien Sereni

N° 5957

Juillet 2007

Thème COM

Rapport de recherche

ISSN 0249-6399 ISRN INRIA/RR--5957--FR+ENG

Circular choosability

Frédéric Havet*, Ross J. Kang†, Tobias Müller‡ , Jean-Sébastien Sereni*

Thème COM — Systèmes communicants
Projet Mascotte

Rapport de recherche n° 5957 — Juillet 2007 — 33 pages

Abstract: In this paper, we study the notion of circular choosability recently introduced by Mohar [8] and Zhu [15]. First, we provide a negative answer to a question of Zhu about circular cliques. We next prove that, for every graph G, $\mathrm{cch}(G) = O\left(\mathrm{ch}(G) + \ln|V(G)|\right)$. We investigate a generalisation of circular choosability, circular f-choosability, when f is a function of the degrees. We also consider the circular choice number of planar graphs. Mohar asked for the value of $\tau := \sup\{\mathrm{cch}(G) : G \text{ is planar}\}$, and we prove that $6 \le \tau \le 8$, thereby providing a negative answer to another question of Mohar. Finally, we study the circular choice number of planar and outerplanar graphs with prescribed girth, and graphs with bounded density.

Key-words: circular choosability, circular choice number, colouring, planar graphs, outerplanar graphs, bounded density

This work was partially supported by Région Provence-Alpes-Côte d'Azur, and the European project IST FET AEOLUS.

* MASCOTTE, I3S (CNRS–UNSA) – INRIA, 2004 Route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex, France. E-mails: {fhavet,sereni}@sophia.inria.fr.

† Oxford University, Department of Statistics, 1 South Parks Road, OX1 3TG, United Kingdom. This author is partially supported by NSERC of Canada and the Commonwealth Scholarship Commission (UK). E-mail: kang@stats.ox.ac.uk.

‡ Oxford University, Department of Statistics, 1 South Parks Road, OX1 3TG, United Kingdom. This author is partially supported by EPSRC, the Department of Statistics, Bekker-la-Bastide fonds, Dr. Hendrik Muller's Vaderlandsch fonds, and Prins Bernhard Cultuurfonds. E-mail: muller@stats.ox.ac.uk.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles, BP 93, 06902 Sophia Antipolis Cedex (France)
Téléphone : +33 4 92 38 77 77 —Télécopie : +33 4 92 38 77 65

Choisissabilité circulaire

Résumé : Nous étudions la notion de choisissabilité circulaire, récemment introduite par Mohar [8] et Zhu [15]. En premier lieu, nous répondons négativement à une question de Zhu à propos des cliques circulaires. Ensuite, nous démontrons que, pour tout graphe G, $\mathrm{cch}(G) = O\left(\mathrm{ch}(G) + \ln |V(G)|\right)$. Puis, nous étudions une généralisation de la choisissabilité circulaire, lorsque la taille de la liste d'un sommet est fonction de son degré. Nous considérons également le nombre de choix circulaire des graphes planaires. Mohar a demandé la valeur de $\tau := \sup\{\mathrm{cch}(G) : G \text{ est planaire}\}$. Nous prouvons que $6 \leq \tau \leq 8$, ce qui fournit une réponse négative à une autre question de Mohar. Enfin, nous étudions le nombre de choix circulaire des graphes planaires et planaires extérieurs de maille donnée, et celui des graphes de densité bornée.

Mots-clés : choisissabilité circulaire, nombre de choix circulaire, coloration, graphes planaires, graphes planaires extérieurs, densité bornée

1 Introduction

Let $G = (V, E)$ be a graph. If p and q are two integers, a (p, q)-*colouring* of G is a function c from V to $\{0, \dots, p-1\}$ such that $q \leq |c(u) - c(v)| \leq p - q$ for any edge $uv \in E$. The circular chromatic number of the graph G is

$$\chi_c(G) = \inf\left\{\frac{p}{q} : G \text{ admits a } (p, q)\text{-colouring}\right\}.$$

For any integer a, we denote by $[a]_q$ the interval $[a - q + 1, a + q - 1]$. Remark that if $uv \in E$ then $c(u) = a \Rightarrow c(v) \notin [a]_q$, where the computations are modulo p.

A *list assignment* L of G is a mapping that assigns to every vertex of v a set of integers, called *colours*. An L-*colouring* of G is a mapping c that assigns to every vertex $v \in V$ a colour $c(v) \in L(v)$. A t-(p, q)-*list-assignment* L is a list assignment such that, for any vertex $v \in V$, $L(v) \subseteq \{0, \dots, p-1\}$ and $|L(v)| \geq tq$.

The graph G is said to be (p, q)-L-*colourable* if there exists a (p, q)-L-*colouring* c, i.e. c is both a (p, q)-colouring and an L-colouring. For any real number $t \geq 1$, it is t-(p, q)-*choosable* if it is (p, q)-L-colourable for any t-(p, q)-list-assignment L. Last, G is *circularly* t-*choosable* if it is t-(p, q)-choosable for any p, q. The *circular list chromatic number* (or *circular choice number*) of G is

$$\mathrm{cch}(G) = \inf\{t \geq 1 : G \text{ is circularly } t\text{-choosable}\}.$$

As shown in [15], $\mathrm{cch}(G) \geq \mathrm{ch}(G) - 1$ for any graph G. By convention, we denote (ordinary) list chromatic number by $\mathrm{ch}(G)$.

Note that this definition of circular choosability differs slightly from the one introduced in [15], in which p is required to be at least $2q$. We have opted for another definition, because the requirement does not seem to make sense. In particular, it contradicts Lemma 7 of [15]: under the definition given in [15], the circular choosability of the single edge K_2 would be one. To see this, suppose that $L_1, L_2 \subseteq \{0, \dots, p-1\}$ are given with $|L_1|, |L_2| \geq (1 + \epsilon)q$ and $p \geq 2q$. It is not difficult to see that there must exist $c_1 \in L_1, c_2 \in L_2$ such that $q \leq |c_1 - c_2| \leq p - q$, so that K_2 would indeed have circular choosability one under the definition from [15]. On the other hand, Lemma 7 of [15] implies that $\mathrm{cch}(K_2) \geq \chi_c(K_2) = 2$.

Moreover, for some graphs G, for t slightly less that the circular choice number, the t-(p, q)-list-assignments L for which G is not (p, q)-L-colourable satisfies that p is not too large compared to q. For example, as in the proof of Proposition 17(ii), one can show that if $t \geq 2$ then a t-(p, q)-list-assignment L for which K_3 is not (p, q)-L-colourable satisfies that $p \leq 4q^2$, whereas $\mathrm{cch}(K_3) = 3$.

We also remark that there is an alternative, continuous definition for circular choosability. Consult [8] or [15] for this definition; however, in the rest of the paper, we will restrict ourselves entirely to *list* (p, q)-*colouring*, the concept we have just outlined here.

The paper is organised as follows: in the first section, we deal with circular cliques (Proposition 2) and even cycles (Corollary 4, Proposition 5 and Conjecture 6). Then, in the second section, we consider the question, given a graph G, of whether $\mathrm{cch}(G)$ is bounded within a constant factor of $\mathrm{ch}(G)$; we answer in the positive when restricted to 2-choosable graphs (Corollary 11), and then use a probabilistic thinning argument to bound the circular choice number of a graph by a constant factor of $(\mathrm{ch}(G) + \ln n)$ where n is the number of vertices (Theorem 12). In the third section, we investigate a generalisation of circular choosability, circular f-choosability, when f is a function of the degrees and attempt to establish an analogue for the result of, independently, Borodin [4] and Erdős, Rubin and Taylor [5] (Theorem 15). The fourth section begins with a question of Mohar [8], who asked for the value of

$$\tau := \sup\{\mathrm{cch}(G) : G \text{ is planar}\}.$$

We shall prove that $6 \leq \tau \leq 8$ (Theorem 24 and Proposition 26). We then give bounds (Proposition 27 and Theorem 35) for the parameter

$$\tau(k) := \sup\{\mathrm{cch}(G) : G \text{ is planar and has girth} \geq k\},$$

and, more generally, the circular choice number of graphs with bounded density (Propositions 28 and 32). For the analogous parameter

$$\tau_o(k) := \sup\{\mathrm{cch}(G) : G \text{ is outerplanar and has girth} \geq k\},$$

we can prove an exact result (Theorem 37).

The four main sections are relatively self-contained and may, with few exceptions, be read independently of each other.

2 Circular cliques and even cycles

We first wish to present a basic tool which will be crucial to several of our results. In the following lemma, G is given, integers $p \geq q$ are given, $t \geq 1$ is also given, as is a t-(p,q)-list assignment L. Furthermore, some vertices $u_1, u_2, \dots, u_k$ are (p,q)-L-precoloured and we are trying to extend the colouring according to some ordering of the vertices $(v_1 = u_1, \dots, v_k = u_k, v_{k+1}, \dots, v_{|V(G)|})$. **Furthermore, we require that the ordering satisfies that every non-precoloured vertex has at most one neighbour with higher index in the order.** We say a colour a in $L(v_j)$ is *extendable* if there exists some (p,q)-L-colouring c of the subgraph induced by $\{v_1, v_2, \dots, v_j\}$ such that $c(v_j) = a$ and c respects the precolouring.

Lemma 1 *Suppose $F = \{w_1, \dots, w_k\}$ is the set of neighbours of v_j with smaller index in the ordering. Let E_i denote the set of extendable colours for w_i, $1 \leq i \leq k$.*

(i) *If w_i has at least $x_i \geq 1$ extendable colours for each i, then v_j has at least $|L(v_j)| - \sum_{i:x_i<2q}(2q - x_i)$ extendable colours.*

(ii) *If w_i has at most $x_i \in \{1, \dots, 2q-1\}$ extendable colours for each i, then v_j has at most $|L(v_j)| - \sum_i (2q - x_i)$ extendable colours if the following holds:*

- *for each i, E_i is contained in some interval $[a_i, \dots, a_i + x_i - 1]$;*

- *the sets* $[a_i]_q \cap [a_i + x_i - 1]_q$ *are mutually disjoint; and*
- $[a_i]_q \cap [a_i + x_i - 1]_q \subseteq L(v_j)$ *for each* i.

Furthermore, the set of extendable colours for v_j *is a subset of* $L(v_j) \setminus \bigcup_i [a_i]_q \cap [a_i + x_i - 1]_q$.

Proof. To determine which colours of v_j are extendable, it suffices to consider all possible colourings of the vertices of F with colours chosen from their respective extendable lists E_i.

(i): By the condition on the ordering of the vertices, a colour of v_j is not extendable for v_j if and only if it is in the intersection $\bigcap_{a \in E_i} [a]_q$ for some i. These intersection sets are maximised when the E_i are intervals, giving $|\bigcap_{a \in E_i} [a]_q| = 2q - x_i$, if $x_i < 2q$, and $|\bigcap_{a \in E_i} [a]_q| = 0$, otherwise.

(ii): Since $x_i < 2q$, it follows from the first condition that $[a_i]_q \cap [a_i + x_i - 1]_q \subseteq \bigcap_{a \in E_i} [a]_q$. Thus, due to the observation made in the proof of (i) together with the last two conditions, the proof holds. □

2.1 Circular cliques

For any two positive integers $a \geq 2b$, the graph $K_{a/b}$, called the *circular clique*, has vertex set $\{0, \ldots, a-1\}$ and ij is an edge if and only if $b \leq |i - j| \leq a - b$. Remark that, for any $k \geq 1$, $K_{(2k+1)/k} \simeq C_{2k+1}$. In [15], Zhu proved that $\mathrm{cch}(C_{2k+1}) = 2 + \frac{1}{k}$ for any $k \geq 1$ and asked whether the circular list chromatic number of $K_{a/b}$ is $\frac{a}{b}$. This is not the case, since the circular cliques contain large complete bipartite subgraphs. More precisely,

Proposition 2 *For any positive integer* N, *there exist positive integers* a, b, *with* $a \geq 2b$, *such that the difference between* $\mathrm{cch}(K_{a/b})$ *and* $\frac{a}{b}$ *is more than* N.

Proof. Let $m \geq \binom{2(N+3)-1}{N+3}$. As is well known, $\mathrm{ch}(K_{m,m}) \geq N+4$ — see [5]. Let $a = 2k+2m$ and $b = k$, for some positive integer $k > 2m$. The graph $K_{a/b}$ contains $K_{m,m}$ as a subgraph: take the vertices $\{0, 1, \ldots, m-1\} \cup \{k+m-1, k+m, \ldots, k+2m\}$. Hence, we have $\mathrm{cch}(K_{a/b}) \geq \mathrm{ch}(K_{a/b}) - 1 \geq \mathrm{ch}(K_{m,m}) - 1 \geq N+3$. However, if $k > 2m$, then $\frac{a}{b} = \frac{2k+2m}{k} < 3$; therefore, $\mathrm{cch}(K_{a/b}) - \frac{a}{b} > N$. □

2.2 Even cycles

As mentioned above, Zhu [15] proved that the circular choice number of the odd cycle C_{2k+1} is $2 + \frac{1}{k}$. The fact that $\mathrm{cch}(C_{2k+1}) \geq 2 + \frac{1}{k}$ is trivial since the circular chromatic number of C_{2k+1} is $2 + \frac{1}{k}$.

We shall use the following lemma later on.

Given a graph G, a *handle* of G is an induced path, with possibly the two endvertices being the same vertex, whose interior vertices all have degree two in G.

Lemma 3 *Fix a positive integer n. Let L be a $\left(2+\frac{2}{n}\right)$-$(p,q)$-list-assignment of the graph G. Suppose $v_0v_1v_2\cdots v_nv_{n+1}$ is a handle H of G. Then any (p,q)-L-precolouring of $G\setminus\{v_1,\dots,v_n\}$ can be extended to the entire graph.*

Proof. Let $t=2+\frac{2}{n}$. Since v_0 has one extendable colour, v_1 has, by Lemma 1(i), at least $tq-2q+1=\frac{2q}{n}+1$ extendable colours. Inductively, we can thus show that v_i has at least $i\frac{2q}{n}+1$ extendable colours, for $i<n$. Now, v_{n+1} has one extendable colour and v_{n-1} has at least $(n-1)\frac{2q}{n}+1$ extendable colours; thus, by Lemma 1(i), v_n has at least $tq-(2q-1)-\left(2q-(n-1)\frac{2q}{n}-1\right)=2$ extendable colours, so that the required colouring indeed exists. □

As a direct corollary of the above lemma, we get:

Corollary 4 *For every integer $n\geq 3$, the cycle C_n has circular choice number at most $2+\frac{2}{n-1}$.*

The upper bound given by Corollary 4 is optimal for odd cycles. But the situation for even cycles seems more complex. Here is a constructive proof of the circular choice number for the smallest even cycle.

Proposition 5 $\mathrm{cch}(C_4)=2$.

Proof. Surely $\mathrm{cch}(C_4)\geq 2$. Let v_1,v_2,v_3 and v_4 denote the vertices of C_4, the vertex v_1 being adjacent to v_2 and v_3. Fix two integers p and q and a 2-(p,q)-list-assignment L. We shall prove that C_4 is (p,q)-L-colourable.

First, remark that if $L(v_1)\cap L(v_4)\neq\emptyset$ or $L(v_2)\cap L(v_3)\neq\emptyset$ then the desired colouring exists: just assign the same colour to, say, v_1 and v_4. Now v_2 and v_3 each have at least $2q-(2q-1)=1$ colour available with distance $>q$ to the colour chosen for v_1 and v_4.

So we suppose now that both intersections are empty.

Let us precolour v_1 with a colour $a\in L(v_1)$. Let $\bar{x}$ be the number colours of $L(v_2)$ that are not extendable, i.e. the number of colours in $L(v_2)\cap[a]_q$. As $L(v_2)\cap L(v_3)=\emptyset$, the number of colours of $L(v_3)$ that are not extendable is at most $2q-1-\bar{x}$.

Now we can apply the lemma once again. Denote by x_i the number of extendable colours of v_i in $L(v_i)$ for $i\in\{2,3,4\}$. The vertex v_4 is adjacent to v_2 and v_3 and since $x_2=2q-\bar{x}$ and $x_3\geq 2q-(2q-\bar{x}-1)=\bar{x}+1$, we infer that $x_4\geq 2q-(2q-(2q-\bar{x}))-(2q-(\bar{x}+1))=1$. □

Conjecture 6 *The circular choice number of every even cycle is two.*

3 Bounds in terms of choosability (and number of vertices)

Recall that the *degeneracy* $\delta^*(G)$ of a graph G is the maximum over all subgraphs of G of the minimum degree. We say that G is *k-degenerate* if $\delta^*(G) \leq k$. The following result was shown in [15]:

Lemma 7 (Zhu, 2005) $\mathrm{cch}(G) \leq 2\delta^*(G)$.

This result was also shown by Zhu [15] to be asymptotically tight by (essentially) showing that the complete bipartite graph K_{k,m^k}, which has degeneracy k, satisfies $\mathrm{cch}(K_{k,m^k}) \geq \left(2 - \frac{2k}{m}\right)k$. This led Zhu to pose the following problem:

Problem 8 *Is there a constant α such that, for every graph G, $\mathrm{cch}(G) \leq \alpha\,\mathrm{ch}(G)$?*

Note that if α exists it is at least two, as $\mathrm{ch}(K_{k,m^k}) \leq \delta^*(K_{k,m^k}) + 1 = k + 1$.

In this section, we first answer in the affirmative to Problem 8 for 2-choosable graphs. In this case, we show that the corresponding constant α is at most $5/2$. Then, more generally, we show that $\mathrm{cch}(G) = O(\mathrm{ch}(G) + \ln(|V(G)|))$, for any graph G.

3.1 2-choosable graphs

The aim of this subsection is to show that every 2-choosable graph has circular choice number at most $5/2$. Therefore, we use the characterisation the 2-choosable graphs by Erdős, Rubin and Taylor [5].

The *heart* of a graph G is the maximal subgraph in which there is no vertex of degree one. Also, the graph consisting of two vertices connected with three independent paths of length i, j and k is denoted by $\theta_{i,j,k}$.

Theorem 9 (Erdős, Rubin and Taylor [5]) *A connected graph is 2-choosable if and only if its heart is either a single vertex, an even cycle or a $\theta_{2m,2,2}$ for some integer $m \geq 1$.*

Proposition 10 *For any $m \geq 1$, $\mathrm{cch}(\theta_{2m,2,2}) \leq 5/2$.*

Proof. Let $xu_1 \dots u_{2m-1}y$, xvy and xwy be the three paths forming $\theta_{2m,2,2}$. Let L be a $5/2$-(p,q)-list-assignment of $\theta_{2m,2,2}$. We prove that $\theta_{2m,2,2}$ admits (p,q)-L-colouring, which is stronger than the desired conclusion.

Assume first that $m = 1$. If $L(u_1) \cap L(v) \cap L(w) \neq \emptyset$, then assign to u_1, v and w the same colour α. Next, assign to x and y a colour in $L(x) \setminus [\alpha]_q$ and a colour in $L(y) \setminus [\alpha]_q$, respectively. This yields the desired colouring. If not, assign to x a colour α in $L(x)$. Let $i_u = |L(u_1) \cap [\alpha]_q|$, $i_v = |L(v) \cap [\alpha]_q|$ and $i_w = |L(w) \cap [\alpha]_q|$. Since $L(u_1) \cap L(v) \cap L(w) \neq \emptyset$, we have $i_u + i_v + i_w \leq 4q - 2$. Thus, u_1, v and w have at least $5q/2 - i_u$, $5q/2 - i_v$ and $5q/2 - i_w$ extendable colours, respectively. Hence, by Lemma 1, y has at least $4q - i_u - i_v - i_w \geq 2$ extendable colours. This proves that $\theta_{2,2,2}$ is (p,q)-L-colourable.

Assume now that $m \geq 2$. If $L(v) \cap L(w) \neq \emptyset$, then assign to v and w the same colour α. Next, assign to x a colour of $L(x) \setminus [\alpha]_q$. By Lemma 1, the vertex u_1 has at least $q/2$ extendable colours, u_2 at least q extendable colours and one can show by induction that u_i has at least $3q/2$ extendable colours, for every $i \in \{3, 4, \ldots, 2m-1\}$. Therefore, again by Lemma 1, y has at least one extendable colour. If $L(v) \cap L(w) = \emptyset$, then assign to x a colour α of its list. Let $i_v = |L(v) \cap [\alpha]_q|$ and $i_w = |L(w) \cap [\alpha]_q|$. Observe that $i_v + i_w \leq 2q-1$, and v and w have at least $5q/2 - i_v$ and $5q/2 - i_w$ extendable colours, respectively. As above, u_1 has at least $q/2$ extendable colours, u_2 at least q, and for every $i \in \{3, 4, \ldots, 2m-1\}$, u_i has at least $3q/2$ extendable colours. Consequently, again by Lemma 1, y has at least $5q/2 - (2(2q - 5q/2) + i_v + i_w) - (2q - 3q/2) \geq 1$ extendable colours. Hence, $\theta_{2m,2,2}$ is (p,q)-L-colourable. □

Corollary 11 *Every 2-choosable graph has circular choice number at most* $5/2$.

Proof. Obviously, it suffices to prove the result for connected graphs. Let L be a $5/2$-(p,q)-list-assignment of G. We prove that every 2-choosable connected graph G admits a (p,q)-L-colouring by induction on x, the difference between the number of vertices of G and the number of vertices of its heart.

Clearly, if $x = 0$, we have the result by Corollary 4 and Proposition 10. Suppose now that $x \geq 1$. Then by the definition of the heart, G has a vertex v of degree one. By induction hypothesis, $G - v$ admits a (p,q)-L-colouring which can be extended to v by Lemma 1. □

3.2 General upper bound

Problem 8 asks whether $\mathrm{cch}(G) = O(\mathrm{ch}(G))$. We are not able to settle the question here, but Theorem 12 below does show that $\mathrm{cch}(G) = O(\mathrm{ch}(G) + \ln(|V(G)|))$. We make no attempt to optimise constants.

Theorem 12 *For any graph G with n vertices, it holds that*

$$\mathrm{cch}(G) \leq 36\,\mathrm{ch}(G) + 54 \ln n + 3.$$

Proof. Fix p, q and set $t = 36\,\mathrm{ch}(G) + 54 \ln n + 3$. Suppose that lists $L(v) \subseteq \mathbb{Z}_p$ of size at least $tq + 1$ are given. If $q = 1$, then we can certainly (p,q)-L-colour G, as $t + 1 > \mathrm{ch}(G)$. So we may assume that $q \geq 2$, without loss of generality.

Let us partition $\left\{0, \ldots, \left\lfloor \frac{p-1}{q-1} \right\rfloor \right\}$ into groups $g_i = \{3i, 3i+1, 3i+2\}$ of three consecutive numbers, where the last group may possibly contain less than three numbers. Out of each group of three, except the very last one, we will pick one element at random, but in such a way that we never pick two consecutive numbers. To be more precise, for $i = 0$ we simply pick one of $0, 1, 2$ uniformly at random. Once a choice has been made for g_{i-1}, we pick one of $3i, 3i+1, 3i+2$ uniformly at random provided we did not choose $3(i-1)+2$ from g_{i-1}. Otherwise, we choose one of $3i+1, 3i+2$ at random each with probability $\frac{1}{2}$. Let us denote the set of selected indices by $\mathcal{K} = \{k : k \text{ was chosen}\}$. With each index $k \in \left\{0, \ldots, \left\lfloor \frac{p-1}{q-1} \right\rfloor\right\}$, we associate an interval $I_k = \{k(q-1), \ldots, (k+1)(q-1)-1\}$ of $\mathbb{Z}_p$. Notice that the I_k are disjoint intervals of length $q-1$. A crucial observation for the sequel is that, if $k, l \in \mathcal{K}$ are distinct and $a \in I_k, b \in I_l$, then $|a-b|_p \geq q$. See Figure 1.

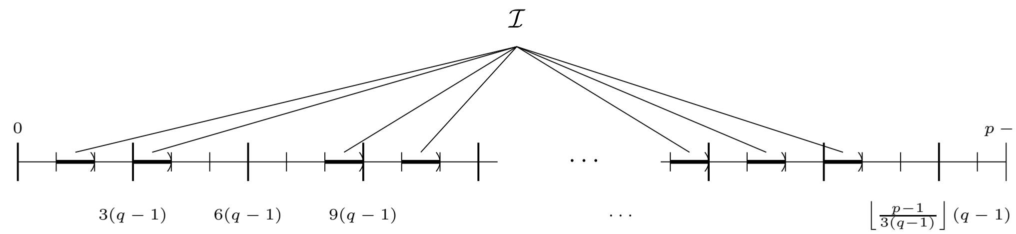

Figure 1: An illustration of the "thinning" procedure for Theorem 12.

Let us set $\mathcal{I} = \bigcup_{k \in \mathcal{K}} I_k$. For each $v \in V$, let us define

$$S(v) = \{k \in \mathcal{K} : I_k \cap L(v) \neq \emptyset\}.$$

The idea for the rest of the proof is to show that t was chosen in such a way that $\mathbb{P}(|S(v)| < \mathrm{ch}(G)) < \frac{1}{n}$ for all v. It will then follow that

$$\mathbb{P}(|S(v)| < \mathrm{ch}(G) \text{ for some } v \in V) < n \cdot \frac{1}{n} = 1.$$

In other words, there must exist a choice of non-adjacent intervals, one from each group of three, for which $|S(v)| \geq \mathrm{ch}(G)$ for all $v \in V$. By the definition of $\mathrm{ch}(G)$, there must exist a colouring c of G with $c(v) \in S(v)$. Let us define a new colouring f by choosing $f(v) \in I_k \cap L(v)$ if $c(v) = k$. This can be done for each v, by the definition of $S(v)$. Now f is a (p,q)-L-colouring, because if $vw \in E(G)$ then $c(v), c(w)$ are distinct elements of $\mathcal{K}$ — consequently, $f(v)$ and $f(w)$ have been chosen from non-adjacent intervals $I_{c(v)}, I_{c(w)}$, so that $|f(v) - f(w)|_p \geq q$.

It remains to show that t is chosen such that $\mathbb{P}(S(v) < \mathrm{ch}(G)) < \frac{1}{n}$ holds. We first claim that the probability that $|S(v)| < \mathrm{ch}(G)$ is bounded above by

$$\mathbb{P}\left(\mathrm{Bi}\left(s, \frac{1}{6}\right) \leq \mathrm{ch}(G)\right),$$

where $s = \left\lceil \frac{t}{3} \right\rceil - 1$. In order to prove the claim, first note that we can "thin" the lists $L(v)$ to get sublists $L'(v) \subseteq L(v)$ with

$$|L'(v)| \geq \left\lceil \frac{|L(v)|}{3(q-1)} \right\rceil - 1 > \frac{t}{3} - 1,$$

and a distance of at least $3(q-1)$ between elements of $L'(v)$. We can construct $L'(v)$ by taking the first, the $(3(q-1)+1)$-th, the $(6(q-1)+1)$-th, and so on up to and including the $((M-1)(q-1)+1)$-th element of $L(v)$, where $M = \left\lceil \frac{tq}{3(q-1)} \right\rceil$, and we discard the $(M(q-1)+1)$-th element, to avoid possible wrap-around effects. Let us write $L'(v) = \{a_1, \dots, a_l\}$ with $a_i \leq a_{i+1}$. For $J \subseteq \{1, \dots, i-1\}$, let $A(i,J)$ denote the event that $a_j \in \mathcal{I}$ for $j \in J$ and $a_j \notin \mathcal{I}$ for all $j \in \{1, \dots, i-1\} \setminus J$. We claim that for every $J \subseteq \{1, \dots, i-1\}$ we have

$$\mathbb{P}(a_i \in \mathcal{I} | A(i,J)) \geq \frac{1}{6}. \tag{1}$$

To this end, note that $a_1, \dots, a_{i-1}$ each give information about the choice made for some group g_l. Also observe that, if $a_i \in I_{3k+1}$ or $a_i \in I_{3k+2}$ for some k, then the probability that a_i is covered by $\mathcal{I}$, given that $A(i,J)$ holds, is at least $\frac{1}{3}$, because, regardless of which element of g_{k-1} was selected, the probability that $3k+1$ (respectively $3k+2$) is selected is at least $\frac{1}{3}$. Now, supposing that $a_i \in I_{3k}$ for some k, then $a_{i-1} \notin I_{3(k-1)+1} \cup I_{3(k-1)+2}$. Therefore, the probability that a_i is covered given that $A(i,J)$ holds is at least the minimum of two probabilities: the probability that $3k$ is chosen given that $3(k-1)$ was chosen from g_{k-1}; and the probability that $3k$ is chosen given that $3(k-1)$ was not chosen from g_{k-1}. It is not hard to see that this minimum is $\frac{1}{6}$, which proves the claim.

A well known bound on the binomial, see for instance [7], states that

$$\mathbb{P}(\mathrm{Bi}(k,p) \leq kp - r) \leq e^{-2\frac{r^2}{k}}.$$

This allows us to write

$$\mathbb{P}(S(v) < \mathrm{ch}(G)) \leq \exp\left[-2\left(\mathrm{ch}(G) - \frac{s}{6}\right)^2 / s\right].$$

So, we would like to have

$$2\left(\mathrm{ch}(G) - \frac{s}{6}\right)^2 > s \ln n,$$

or in other words

$$\frac{s^2}{36} - \left(\frac{\mathrm{ch}(G)}{3} + \frac{\ln n}{2}\right) s + \mathrm{ch}^2(G) > 0.$$

This is certainly true if

$$s > \frac{\frac{\mathrm{ch}(G)}{3} + \frac{\ln n}{2} + \sqrt{\left(\frac{\mathrm{ch}(G)}{3} + \frac{\ln n}{2}\right)^2 - 4\,\mathrm{ch}(G)^2 \frac{1}{36}}}{\frac{2}{36}},$$

and this in turn is certainly true if

$$\frac{t}{3} - 1 \geq 12\,\mathrm{ch}(G) + 18 \ln n,$$

i.e. if $t \geq 36\,\mathrm{ch}(G) + 54 \ln n + 3$. □

Recall that Alon [1] has shown that $\mathrm{ch}(G) = \Omega(\ln d)$, where d denotes the average degree of G. Theorem 12 thus shows that the existence of such a constant α for Problem 8 can only be disproved by considering sparse graphs; there is a choice of $\alpha = \alpha(\epsilon)$ that works for all graphs with average degree at least ϵn.

We also remark that it is straightforward to give an upper bound for cch in terms of ch that is exponential in ch: the mentioned result of Alon also shows that $\mathrm{ch}(G) = \Omega(\ln(\delta^*(G)))$ — as there is some subgraph of G with minimum degree $\delta^*(G)$, this subgraph certainly has average degree at least $\delta^*(G)$. On the other hand, $\mathrm{cch}(G) \leq 2\delta^*(G)$, so that

$$\mathrm{cch}(G) \leq e^{\beta\,\mathrm{ch}(G)},$$

for some $\beta > 0$.

3.3 Complete multipartite graphs

Complete bipartite graphs are a natural class to consider, being the canonical examples of graphs with low chromatic number, yet high choosability. The following adaptation of our argument sharpens the general upper bound given by Theorem 12 in the special case when $G = K_{r*m} = K_{\underbrace{m,\dots,m}_{r}}$ is the balanced, complete r-partite graph, with each part of size m. Recent work of Gazit and Krivelevich [6] shows that $\mathrm{ch}(K_{r*m}) = (1+o(1))\frac{\ln m}{\ln\left(1+\frac{1}{r-1}\right)}$, so that the bound on $\mathrm{cch}(K_{r*m})$ given by Proposition 13 is indeed sharper than the general upper bound given by Theorem 12. The work of Gazit and Krivelevich also considers complete multipartite graphs with not all parts of equal size, but with not too much difference in size between the smallest and the largest parts. In principle, our result and proof can be adapted to also cover this additional class of complete multipartite graphs, but we have chosen to omit this here. It should be noted that Alon and Zaks [3] prove a very similar result to Proposition 13 below for T-choosability of complete bipartite graphs (and in fact give a similar proof).

Proposition 13 $\mathrm{cch}(K_{r*m}) \leq \frac{3(\ln m+\ln r)}{\ln\left(1+\frac{1}{6r-1}\right)} + 1.$

Proof. We proceed as in the proof of Theorem 12, selecting a subset $\mathcal{K} \subseteq \left\{0,\dots,\left\lfloor\frac{p-1}{q-1}\right\rfloor\right\}$ of indices, no two consecutive, at random in the same way as in the proof of Theorem 12. We shall now partition the indices that have been selected into r sets $\mathcal{K}_1,\dots,\mathcal{K}_r$, by assigning each index $k \in \mathcal{K}$ uniformly at random to one of $\mathcal{K}_1,\dots,\mathcal{K}_r$, independently from all other elements in $\mathcal{K}$. Let us write $\mathcal{I}_1 = \bigcup_{k\in\mathcal{K}_1} I_k,\dots,\mathcal{I}_r = \bigcup_{k\in\mathcal{K}_r} I_k$.

If the partition of the vertex-set is $V = V_1 \uplus \cdots \uplus V_r$, then we will attempt to colour K_{r*m} using only the colours from $\mathcal{I}_1$ for the vertices of V_1, only the colours from $\mathcal{I}_2$ for the vertices of V_2, and so on. This can be done provided that, for each $i \in \{1,2,\dots,r\}$ and each $v \in V_i$, we have

$$L'(v) = L(v) \cap \mathcal{I}_i \neq \emptyset. \tag{2}$$

To see this, notice that if (2) holds then we can define a colouring by picking an arbitrary $f(v) \in L'(v)$ for each $v \in V$. We get a proper (p,q)-L-colouring in this way, because if $vw \in E(K_{r*m})$ then $f(v)$ and $f(w)$ will have been chosen from distinct — and thus also non-adjacent — intervals as it must be the case that $(v,w) \in V_i \times V_j$ for some $i \neq j$ and $\mathcal{K}_i, \mathcal{K}_j$ are disjoint. Because $f(v), f(w)$ have been chosen from non-adjacent intervals, their distance is at least q in $\mathbb{Z}_p$.

So we see that it suffices to show that $\mathbb{P}(L'(v) = \emptyset) < \frac{1}{rm}$. We can again take $a_1 \leq \cdots \leq a_l$ in $L(v)$ with a distance of at least $3(q-1)$ between them and $l > \frac{t}{3} - 1$. The probability that $a_j \in \mathcal{I}_i$ given that $a_1, \ldots, a_{j-1} \notin \mathcal{I}_i$ is at least $\frac{1}{6r}$ (by an argument analogous to the argument used the proof of Theorem 12), so that

$$\mathbb{P}(L'(v) = \emptyset) < \left(1 - \frac{1}{6r}\right)^{\frac{t}{3}-1}$$

and it is clear that $t \geq \frac{3(\ln m + \ln r)}{\ln\left(1 + \frac{1}{6r-1}\right)} + 1$ will give that $\mathbb{P}(L'(v) = \emptyset) < \frac{1}{rm}$, which concludes the proof. □

4 Circular f-choosability

A graph G is ***degree-choosable*** if, given any list-assignment L such that every has a list of size at least its degree, there exists a proper L-colouring of G. A theorem independently proved by Borodin [4] and Erdős, Rubin and Taylor [5] — see also [12] — states that a connected graph is degree-choosable, unless it is a *Gallai tree*, i.e. unless each of its blocks is complete or an odd cycle. In this section, we study the analogous problem for circular choosability. This result is interesting in itself but can also be used as a tool to extend list circular colouring from a graph to larger graphs. See Lemma 31 for an example.

Let G be a graph and f a weight function on the vertices, i.e. $f : V(G) \to \mathbb{R}^+$. An f-(p,q)-***list-assignment*** L is a list assignment such that for any vertex $v \in V$, $L(v) \subseteq \{0, \ldots, p-1\}$ and $|L(v)| \geq \max\{1, f(v)q\}$. For any real number $t \geq 1$, G is f-(p,q)-***choosable*** if it is (p,q)-L-colourable for any f-(p,q)-list-assignment L. Last, G is ***circularly*** f-***choosable*** if it is f-(p,q)-choosable for any p, q. The condition $|L(v)| \geq \max\{1, f(v)q\}$ may seem unnatural but is just here to allow some vertices v to have $f(v) = 0$.

It is easy to see that every graph is circularly $2d$-choosable. This is best possible since $\mathrm{cch}(K_2) = 2$; however, the edge is essentially the unique graph attaining the bound. We next consider $(2d-1)$-choosability.

Let G be a graph and v a vertex. A v-***greedy*** ordering of G is an ordering $(v_1, v_2, \ldots, v_n = v)$ of the vertices such that for every $i \in \{1, 2, \ldots, n-1\}$, v_i has a neighbour in $\{v_{i+1}, \ldots, v_n\}$. It is easy to see that if v is a vertex of a connected graph G then there is a v-greedy ordering of G.

Proposition 14 *Every connected graph with $n \geq 3$ vertices is circularly $(2d-1)$-choosable.*

Proof. It suffices to prove it for connected graphs. Let G be a connected graph and v a vertex with degree at least two. Consider a v-greedy ordering $(v_1, v_2, \ldots, v_n = v)$ of G. Note that $v_{n-1}v$ is an edge. Colour greedily the vertices v_i for $i \in \{1, 2, \ldots, n-2\}$; this is possible since, at each step, v_i has at least $(2d(v_i)-1)q - (d(v_i)-1)(2q-1) = q + d(v_i) - 1$ colours available. Then, by Lemma 1, v_{n-1} has at least $q + d(v_{n-1}) - 1$ extendable colours. Therefore, by Lemma 1, v_n has at least one extendable colour. Hence, one can extend the colouring to both v_{n-1} and v. □

Proposition 14 is best possible since the circular choice number of K_3 is three.

4.1 Circular $(2d-2)$-choosability

The main result of this section is as follows:

Theorem 15 *Let G be a connected graph on $n \geq 3$ vertices.*

(i) *G is circularly $(2d-2)$-choosable if:*

  (a) *it is 2-connected and not a cycle of length different from four; or*

  (b) *it is not 2-connected and one of its blocks is neither an edge nor a cycle of length at least five.*

(ii) *G is not $(2d-2)$-choosable if:*

  (a) *it is a tree; or*

  (b) *it is an odd cycle.*

We start by giving a lemma which allows us to extend the property of a graph H being $(2d-2)$-choosable to graphs which contain H as an induced subgraph.

Lemma 16 *Let G be a connected graph. If G has an induced subgraph H which is circularly $(2d_H-2)$-choosable and contains at least one edge, then G is also circularly $(2d_G-2)$-choosable.*

Proof. Let us prove it by induction on $|V(G)| - |V(H)|$, the result holding trivially if $|V(G)| - |V(H)| = 0$. Suppose now that $|V(G)| > |V(H)|$. Since G is connected, there exists a set of disjoint non-trivial trees T_i, $1 \leq i \leq l$, each with a unique vertex r_i in H such that $V(H) \cup \bigcup_{i=1}^{l} V(T_i) = V(G)$. Let x be a leaf of T_1 distinct from r_1. Then $G - x$ is connected and contains H as an induced subgraph.

Let L be a $(2d_G-2)$-(p,q)-list assignment of G. Assign a colour a to x. Let L' be the list-assignment defined by $L'(v) = L(v) \setminus [a]_q$ if v is a neighbour of x and $L'(v) = L(v)$ otherwise. Clearly, L' is a $(2d_{G-x}-2)$-(p,q)-list assignment of $G-x$. So, by induction, it admits a circular (p,q)-L-colouring c which can trivially be extended to x setting $c(x) = a$.
□

We will now show circular $(2d-2)$-choosability for some small important graphs. Recall from Subsection 3.1 the definition of $\theta_{i,j,k}$; as we will observe later, these graphs are important for 2-connected graphs. We will also consider the *flag*, the graph with vertex set $\{t, u, v, w\}$ and edge set $\{tu, tv, uv, vw\}$.

Proposition 17 *The following graphs are circularly* $(2d-2)$*-choosable:*

(i) $\theta_{i,j,k}$, *and*

(ii) *the flag.*

Proof of part (i) **of Proposition 17**. Let $xu_1 \dots u_{i-1}y$, $xv_1 \dots v_{j-1}y$ and $xw_1 \dots w_{k-1}y$ be the three paths forming $\theta_{i,j,k}$. Since there are no multiple edges, we may assume that $i, j \geq 2$.

Let L be a $(2d-2)$-(p,q)-list assignment of $\theta_{i,j,k}$. Every vertex has a list of size $2q$ except x and y which have lists of size $4q$.

Suppose that $L(u_1) \cap L(v_1) \neq \emptyset$, and colour u_1 and v_1 with the same colour. Then, extend greedily this algorithm according to the ordering $(u_1, \dots, u_{i-1}, v_1, \dots, v_{j-1}, y, w_{k-1}, \dots, w_1, x)$.

Hence, we now assume that $L(u_1) \cap L(v_1) = \emptyset$ are disjoint. Assign to x a colour c in its list. Let α and β be the number of extendable colours of u_1 and v_1, respectively. Observe that $\alpha + \beta \geq 4q - (2q-1) = 2q+1$. Now, by Lemma 1, one can extend the colouring to $\theta_{i,j,k} - y$ such that u_{i-1}, v_{j-1} and w_{k-1} have at least α, β and one extendable colours, respectively. Then, y has at least $4q - 6q + \alpha + \beta + 1 \geq 2$ extendable colours. □

Before the proof of the next part, we note the following easy consequence of Lemma 1:

Lemma 18 *If* L *is a* (p,q)*-list assignment of* K_2 *with* $p \geq 2q$ *such that* $|L(x)| \geq q$ *and* $|L(y)| \geq q+1$*, then* K_2 *is* (p,q)*-*L*-colourable.*

Proof of part (ii) **of Proposition 17**. Suppose for a contradiction that there exists a $(2d-2)$-(p,q)-list assignment L such that the flag is not (p,q)-L-colourable. Without loss of generality, we may assume that $L(w) = \{0\}, |L(v)| = 4q$ and $|L(u)| = |L(t)| = 2q$.

Claim 1 $0 \notin L(t) \cup L(u)$

Suppose not. By symmetry, we may assume that $0 \in L(t)$. Then, there exists $k_u \in L(u) \setminus [0]_q$ and $k_v \in L(v) \setminus ([0]_q \cup [k_u]_q)$. Hence, $c(t) = 0, c(u) = k_u, c(v) = k_v, c(w) = 0$ is a (p,q)-L-colouring of the flag, a contradiction. This proves Claim 1.

Claim 2 $(L(t) \cup L(u)) \cap [0]_q \neq \emptyset$.

Suppose for a contradiction that $(L(t) \cup L(u)) \cap [0]_q = \emptyset$. Let k_t, k_u, and k_v be the smallest integers in $L(t)$, $L(u)$, and $L(v) \setminus [0]_q$, respectively. By symmetry, we may assume that $k_t \leq k_u$. If $k_t \leq k_v$, then $|L(v) \setminus ([0]_q \cup [k_t]_q)| \geq q+1$ and $|L(u) \setminus [k_t]_q| \geq q$. So, by Lemma 18, the colouring $c(t) = k_t, c(w) = 0$ may be extended to a (p,q)-L-colouring of the flag, a contradiction. If $k_v < k_t$, then $|L(t) \setminus [k_v]_q| \geq q+1$ and $|L(u) \setminus [k_v]_q| \geq q+1$. So, by Lemma 18, the colouring $c(v) = k_v, c(w) = 0$ may be extended to a (p,q)-L-colouring of the flag, a contradiction. This proves Claim 2.

By symmetry, we may assume that $(L(t) \cup L(u)) \cap [0, q-1] \neq \emptyset$. Let k_t be the minimum integer in this set. Without loss of generality, we may assume that $k_t \in L(t)$.

Claim 3 $|L(u) \cap [k_t]_q| \geq q+1$.

Since $|L(v) \setminus ([0]_q \cup [k_t]_q)| \geq q+2$, it follows that $|L(u) \setminus [k_t]_q| < q$; otherwise, by Lemma 18, the colouring $c(t) = k_t, c(w) = 0$ may be extended to a (p,q)-L-colouring of the flag, a contradiction. Thus, $|L(u) \cap [k_t]_q| \geq q+1$. This proves Claim 3.

Hence, there exist l_u and $k_u \geq l_u + q$ both in $L(u) \cap [k_t]_q$. By minimality of k_t, $l_u \in [k_t - q + 1, 0] \cap L(u)$.

Analogously to Claim 3, $|L(t) \cap [l_u]_q| \geq q+1$. Hence, there exists $l_t \in L(t) \cap [l_u - q + 1, l_u]$. Now, $J = [l_t]_q \cup [k_u]_q \cup [0]_q \subset [l_t - q + 1, k_u + q - 1]$. But $l_t \geq l_u - q + 1$ and $k_u - l_u \geq q$, so $|J| < 4q$. It follows that there exists $l_v \in L(v) \setminus J$. Then $c(t) = l_t, c(u) = k_u, c(v) = l_v, c(w) = 0$ is a (p,q)-L-colouring of the flag, a contradiction. □

Corollary 19 *Let G be a 2-connected graph which is not a cycle. Then G is circularly $(2d-2)$-choosable.*

Proof. If G contains a complete graph K_n, $n \geq 4$, then, by Lemma 16, G is circularly $(2d-2)$-choosable since $\mathrm{cch}(K_n) = n \leq 2n-2$. So assume that G is neither a clique nor a cycle. It suffices to show that G has an induced $\theta_{i,j,k}$ for some i, j, k.

Let us first show that there exist three vertices u, v and w such that $uv \in E(G)$, $vw \in E(G)$ and $uw \notin E(G)$. Since G is not complete, there exist two vertices u and u' which are not linked by an edge. Since G is connected, there is a path between u and u'. Let P be a shortest (u,u')-path and let v and w be respectively the second and third vertices on P. Then uv and vw are edges of the paths and uw is not an edge otherwise it would shortcut P.

Now, since G is 2-connected, there is a (u,w)-path in $G - v$. Let $P = u_0 u_1, \dots u_{l-1} u_l$ with $u_0 = u$ and $u_l = w$ be a shortest such path.

Suppose first that there is an edge vu_i for some $i \in \{1, 2, \dots, l-1\}$. Let i_1 and i_2 be the smallest and second smallest integers $i > 0$ such that vu_i is an edge. Then the graph induced by $\{v, u_0, \dots, u_{i_2}\}$ is a $\theta_{1, i_1+1, i_2-i_1+1}$.

Suppose now that vu_i is not an edge for all $i \in \{1, 2, \dots, l-1\}$. Then $C = vu_0 \dots u_l v$ is an induced cycle of length at least 4. Since G is not a cycle and G is connected, there is a C-path in G — that is, a path with both ends distinct and in $V(C)$, and the internal vertices in $V(G) \setminus V(C)$. Let $P = x_0 \dots x_m x_{m+1}$ be such a path with minimal length. Without loss of generality, we may assume that $v = x_0$. By minimality of P, P is an induced path and the sole internal vertices that are adjacent to a vertex of C are x_1 and x_m.

Assume first that x_1 has at least three neighbours in C. Let j_1 and j_2 be the smallest and second smallest integers $j \geq 0$ such that vu_j is an edge. By rotation around C, we may assume that $j_2 \neq l$. Then the graph induced by $\{x_1, v, u_0, \dots, u_{j_2}\}$ is a $\theta_{1,j_1+2,j_2-j_1+1}$.

If x_1 has exactly two neighbours in C, v and u_i for some $1 \leq i \leq l$, then the graph induced by $\{x_1\} \cup V(C)$ is a $\theta_{1,i+2,l-i+2}$.

Similarly, we get the result if x_m has at least two neighbours in C.

Hence, we may assume that x_1 and x_m have both a unique neighbour in C. It follows that $C + P$ induces a $\theta_{i,j,k}$ for some i, j, k. □

Corollary 20 *Let G be a connected graph. If G contains a flag (not necessarily induced) then G is circularly $(2d-2)$-choosable.*

Proof. If G contains a flag, then it contains as an induced subgraph either the flag, L_4 — the graph obtained from K_4 by deleting on edge — or K_4 as an induced subgraph. By Lemma 16, it suffices to prove that these three graphs are circularly $(2d-2)$-choosable. Since the circular choice number of K_4 is four, K_4 is circularly $(2d-2)$-choosable. Note that $L_4 = \theta_{1,2,2}$. Thus, L_4 and the flag are circularly $(2d-2)$-choosable by Proposition 17. □

Proposition 21 *No tree is circularly $(2d-2)$-choosable*

Proof. We show the following by induction on the number of vertices of a tree T: for any integer k, there exists $q_k(T)$ such that for any $q \geq q_k(T)$ and $p \geq \Delta(T)q + 4kq$, there exists a (p,q)-list-assignment L such that (i) for any $v \in V(T)$, $L(v) \geq \max\{(2d(v)-2)q, k\}$ and (ii) T is not (p,q)-L-colourable.

The results holds trivially for a tree with two vertices. It suffices to take $q_k = k+1$, $L(a) = L(b) = \{1, \dots, k\}$ and $p \geq q + 4kq$.

Let T be a tree with $n \geq 2$ vertices and let $k \geq 1$.

Let u be vertex of T such that at most one of its neighbour is not a leaf — such a vertex exists: consider the second vertex of a longest path in T. Let $u_0, u_1, \dots, u_i$ be the neighbours of u with u_j leaves for $j \in \{1, 2, \dots, k\}$. Then, $T' = T - \{u_1, \dots, u_i\}$ is a tree.

Let $k' = k \times i$. By induction hypothesis, there exists $q'_k(T')$ fulfilling the property of the theorem.

Let $q_k(T) = \max\{k, q'_k(T')\}$. Let $q \geq q_k(T)$ and $p \geq \Delta(T)q + 4kq$. By induction, there exists a (p,q)-list-assignment L' of T' such that (i) for any $v \in V(T')$, $L'(v) \geq \max\{(2d_{T'}(v)-2)q, k'\}$ and (ii) T' is not (p,q)-L'-colourable.

For any $j \in \{1, 2, \dots, i\}$, let I_j be a set of k consecutive integers such that the sets J_j of $2q-k$ colours that I_j forbids are pairwise disjoint and do not intersect $L'(u)$. This is possible since p is large enough. Now, set $L(u_j) = I_j$ and $L(u) \subset L'(u) \cup \bigcup_{j=1}^{i} J_j$.

One easily sees that L is the desired assignment. Indeed, if there were a (p,q)-L-colouring of T, then each u_j must be assigned a colour in I_j and so u cannot receive a colour in J_j. Thus it receives a colour in $L'(u)$. But then c is a (p,q)-L'-colouring of T', a contradiction. □

We are now ready to prove the main theorem.

Proof of Theorem 15. Part (ii) of the theorem follows from Proposition 21 and the fact that $\mathrm{cch}\,(C_{2k+1}) = 2 + \frac{1}{k} > 2$. Part (i)(a) is just Corollary 19. In part (i)(b), since a block is also an induced subgraph of G, we can just apply Lemma 16 and one of Proposition 5 or Corollary 20. □

For a few graphs, we could not determine whether they are circularly $(2d-2)$-choosable or not. We conjecture the following:

Conjecture 22 *A connected graph is circularly* $(2d-2)$*-choosable unless it is a tree or an odd cycle.*

This conjecture would be true if Conjecture 6 holds and every graph consisting of a cycle of length at least 5 and an edge with one endvertex in the cycle is $(2d-2)$-choosable.

Despite not completely characterising the circularly $(2d-2)$-choosable graphs, we can characterise such graphs that have a universal vertex. This will be useful in the next section. Recall that a vertex v of a graph G is said to be *universal* (in G) if it is adjacent to every other vertex.

Corollary 23 *Let* G *be a graph with a universal vertex. Then* G *is* $(2d-2)$*-choosable unless it is* K_3 *or a star.*

5 Planar graphs and graphs of bounded density

5.1 Planar graphs

Mohar [8] asked for the value of $\tau := \sup\{\mathrm{cch}(G) : G \text{ is planar}\}$. We first show that every planar graph is circularly 8-choosable, and so $\tau \leq 8$. Afterwards, we exhibit for each $n \geq 2$ a planar graph whose circular choice number is at least $6 - \frac{1}{n}$, implying $\tau \geq 6$.

The proof of the following theorem is inspired by Thomassen's proof for 5-choosability of planar graphs [11]. Indeed, it can be considered to be a generalisation of his celebrated result (if we let $q = 1$ in Proposition 25).

Theorem 24 *Every planar graph is circularly* 8*-choosable.*

We actually establish the following stronger result:

Proposition 25 *Let* G *be a near triangulation, i.e. a simple planar graph which consists of a cycle* C *and vertices and edges inside* C *such that each bounded face is bounded by a triangle. Fix* $p \geq q$ *two integers, and let* L *be a* (p,q)*-list-assignment such that* $\forall v \in V, L(v) \subseteq \{0, \dots, p-1\}$ *with* $|L(v)| \geq 4q-1$ *if* $v \in C$ *and* $|L(v)| \geq 8q-3$ *otherwise. Then any* (p,q)*-*L*-precolouring of two adjacent vertices of* C *can be extended to a* (p,q)*-*L*-colouring of* G*.*

Proof. The proof is by induction on the number of vertices n. The result holds if G is a triangle since we have at least $4q-1-2(2q-1)=1$ choice to colour the last vertex. Suppose now that the result is true for every near triangulation with at most $n-1 \geq 3$ vertices, and let G be a near triangulation with n vertices. We denote by $u_1u_2 \dots u_k$ the outer cycle of G, and let u_1 and u_2 be the two precoloured vertices.

First Case: G has a chord $u_iu_j, i<j$. Then we use the induction hypothesis on the near triangulation whose outer cycle is $u_1u_2 \dots u_iu_ju_{j+1}, \dots, u_k, u_1$. Next we use the induction hypothesis on the near triangulation whose outer cycle is $u_iu_{i+1} \dots u_ju_i$, the two precoloured vertices being u_i and u_j.

Second Case: G has no chord. Let $v_1, \dots, v_d$ be the neighbours of u_k that do not belong to C. Without loss of generality, we can assume that $u_{k-1}v_1v_2 \dots v_du_1$ is a path. Let a and b be two colours of $L(u_k) \setminus [c(u_1)]_q$ such that $[a]_q \cap [b]_q = \emptyset$. Such colours exist since $|L(u_k)| \geq 4q-1 \geq 2q+2q-1$. We consider the graph G' obtained from G by removing the vertex u_k. It is a near triangulation with outer cycle $u_1u_2 \dots u_{k-1}v_1v_2 \dots v_du_1$. We define the list-assignment L' of G' by $L'(v) = L(v)$ if $v \notin \{v_1, v_2, \dots, v_d\}$ and $L'(v) = L(v) \setminus ([a]_q \cup [b]_q)$ otherwise. We can then use the induction hypothesis on G' and L' (since $\forall i, |L'(v_i)| \geq 8q-3-2(2q-1) = 4q-1$). Now we complete the colouring of G by colouring u_k with a if $c(u_{k-1}) \notin [a]_q$ and with b otherwise. □

Proposition 26 *For any $n \geq 2$, there exists a planar graph G_n with* $\mathrm{cch}(G_n) \geq 6 - \frac{1}{n}$.

Proof. Let $t = 6 - \frac{1}{n}$, $n \geq 2$. Let p and q be two integers with $q = 3n$, and p much larger than $tq = 18n-3$. All the computations and intervals are modulo p. We consider the planar graph H_m of Figure 2, with $m = 2q-1$. We construct the graph G_n by taking $(tq)^2$ copies of H_m and identifying the vertices u of each copy, and identifying the vertices v of each copy. Let us first assign the lists $L(u) = [r, r+tq-1]$ to u and $L(v) = [s, s+tq-1]$ to v with r and s such that $[r-q+1, r+tq+q-1] \cap [s-q+1, s+tq+q-1] = \emptyset$. Now, for each $(a,b) \in L(u) \times L(v)$, we shall assign lists to the vertices of the copy $H_{a,b}$ of H_m in such a way that if u is coloured a and v is coloured b, then the subgraph $H_{a,b}$ cannot be (p,q)-L-coloured. So fix a copy $H_{a,b}$ of H_m. We define the following list assignment: For any $i \in \{0, 1, \dots, m\}$, $L(x_i) = [a]_q \cup [b]_q \cup I_i \cup J_i$ where

$$J_i = [c_i, c_i + 2q - 2 - i]$$

$$I_i = [c_{i-1} + q - i, c_{i-1} + q - 1] = [c_{i-1}]_q \cap [c_{i-1} + 2q - 1 - i]_q$$

(note that $I_0 = J_m = \emptyset$) and the constants c_i, for $i \in \{0, 1, \dots, m\}$, are chosen in such a way that any two intervals are at least circular distance $2q$ apart from each other (except if one is J_i and the other is I_{i+1}, $0 \leq i \leq m-1$). Therefore, each list has size $|L(x_i)| = 2(2q-1) + i + (2q-1-i) = 6q-3 = 18n-3 = tq$.

Let us try to (p,q)-L-colour G_n. We may assume without loss of generality that u is precoloured a and v is precoloured b. Let us examine the subgraph $H_{a,b}$. Clearly, the set of extendable colours for x_0 is a subset of the interval J_0. By Lemma 1(ii), the set

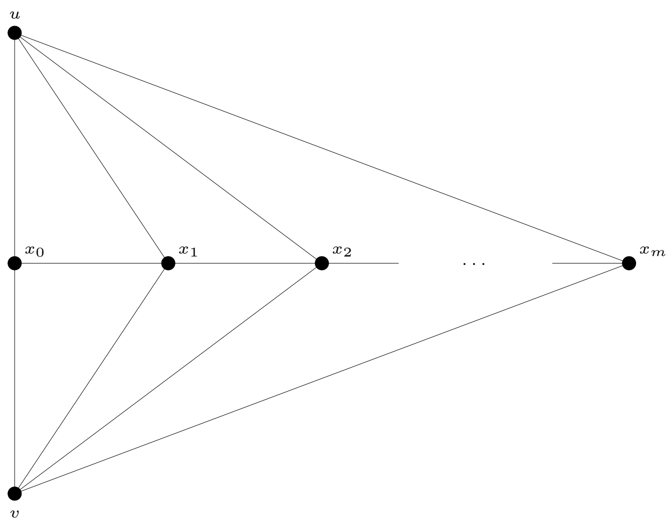

Figure 2: The graph H_m.

of extendable colours for x_1 is a subset of the interval J_1. We can now iteratively apply Lemma 1(ii) to show that the set of extendable colours for x_i is a subset of the interval J_i for all $i \in \{1, \ldots, m\}$. However, $J_m = \emptyset$ and hence the graph is not (p, q)-L-colourable. $\square$

5.2 Lower bounds for planar graphs with prescribed girth

A natural question is to ask what happens when we restrict ourselves to planar graphs with high girth. The *girth* of a graph G is the size of a smallest cycle of G. We study the circular choice number of planar graphs with girth at least k. For any $k \geq 3$, we define

$$\tau(k) := \sup\{\mathrm{cch}(G) : G \text{ is planar and has girth} \geq k\}.$$

As $\tau = \tau(3)$, $6 \leq \tau(3) \leq 8$. In this section and the next, we will establish the following bounds:

- $\tau(k) \geq 2 + \frac{4}{k-2}$ for any $k \geq 3$;
- $\tau(4) \leq 6$; $\tau(5) \leq 4 + \frac{1}{2}$; $\tau(6) \leq 4$; $\tau(8) \leq 3 + \frac{1}{3}$; $\tau(9) \leq 3$; and
- $\tau(4l + 2) \leq 2 + \frac{2}{l}$ for any $l \geq 1$.

First, we wish to exhibit examples to bound $\tau(k)$ from below.

Proposition 27 *For any $k \geq 3$ and $n \geq 2$, there exists a planar graph $G_{k,n}$ of girth k with* $\mathrm{cch}(G_{k,n}) \geq 2 + \frac{4}{k-2} - \frac{1}{n}$.

Proof. We may assume $k \geq 4$ due to Proposition 26. Let $t = 2 + \frac{4}{k-2} - \frac{1}{n}$, $n \geq 2$. Let p and q be two integers with $q = (k-2)n$, and p much larger than $tq = 2kn - (k-2)$. All the computations and intervals are modulo p.

We will have to consider separate cases for the parity of k. If $k = 2l+1$, we construct the planar graph H'_m from H_m of Figure 2 by subdividing each of the edges ux_i and vx_i exactly $(l-1)$ times; see Figure 3. Denote the vertices on the induced path of length l between u and x_i by $u_i^1, \ldots, u_i^{l-1}$, and the vertices on the path between v and x_i by $v_i^1, \ldots, v_i^{l-1}$. If $k = 2l+2$, we will further subdivide the edge $x_i x_{i+1}$ (once). The calculations for the two cases are similar, so from now on we will only consider the case $k = 2l+1$.

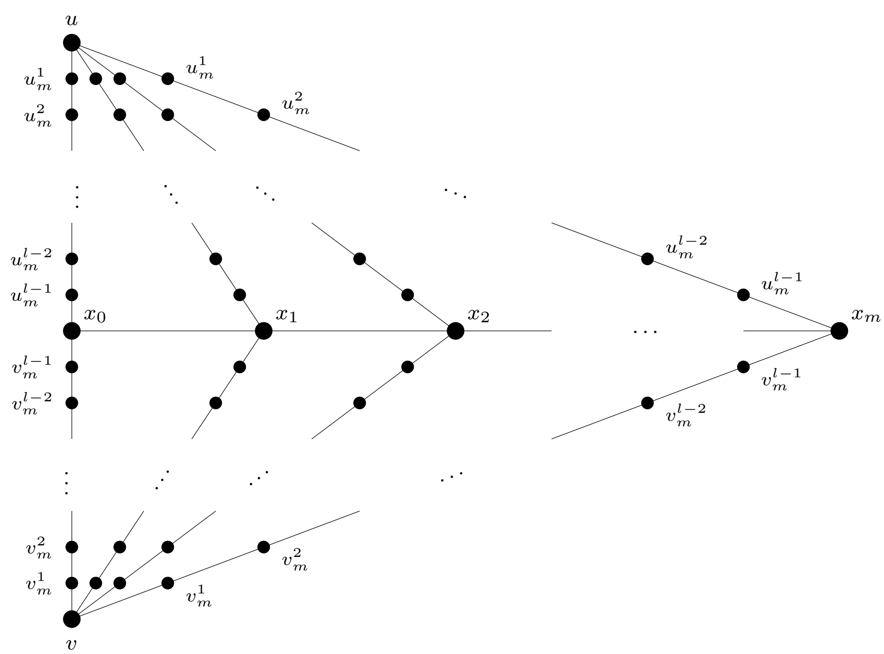

Figure 3: The graph H'_m for the case $k = 2l+1$.

Similarly to Proposition 26, the graph $G_{k,n}$ is constructed from $(tq)^2$ copies of H'_m by identifying the vertices u of each copy, and identifying the vertices v of each copy. Clearly, such a graph has girth exactly k. Again, we assign the lists $L(u) = [r, r+tq-1]$ to u and $L(v) = [s, s+tq-1]$ to v with r and s chosen so that the lists $L(u)$ and $L(v)$ are at least circular distance $2q$ away from each other. For each $(a,b) \in L(u) \times L(v)$, we shall assign lists to the vertices of the copy $H_{a,b}$ of H'_m in such a way that if u is coloured a and v is coloured b, then the subgraph $H_{a,b}$ cannot be (p,q)-L-coloured. We fix a copy $H_{a,b}$ of H'_m.

We define the following list assignment: For any $0 \leq i \leq m$ and $1 \leq j \leq l-1$, $L(u_i^j) = I_{i,j}^u \cup J_{i,j}^u$ and $L(v_i^j) = I_{i,j}^v \cup J_{i,j}^v$ where

$$J_{i,j}^u = [c_{i,j}^u, c_{i,j}^u + j(tq-2q)] \text{ and}$$

$$I_{i,j}^u = [c_{i,j-1}^u + (j-1)(tq-2q) - q + 1, c_{i,j-1}^u + q - 1] = [c_{i,j-1}^u]_q \cap [c_{i,j-1}^u + (j-1)(tq-2q)]_q,$$

and the $J_{i,j}^v, I_{i,j}^v$ are defined analogously in terms of the constants $c_{i,j}^v$. Here $c_{i,0}^u = a, c_{i,0}^v = b$, and the constants $c_{i,j}^w$, for $w \in \{u,v\}, 0 \leq i \leq m$ and $0 \leq j \leq l-1$, are chosen sufficiently far apart from each other. Now, each of these lists has size $|L(u_i^j)| = |L(v_i^j)| = (j(tq-2q)+1) + (2q-1-(j-1)(tq-2q)) = tq$. We have yet to assign lists to the vertices x_i. For any $0 \leq i \leq m$, $L(x_i) = I_{i,l}^u \cup I_{i,l}^v \cup I_i \cup J_i$ where $I_{i,l}^u$ and $I_{i,l}^u$ as above and

$$\begin{aligned}
J_i &= [c_i, c_i + tq - 2(2q-(l-1)(tq-2q)-1) - 1 - i(2(l-1)(tq-2q)+2)] \\
&= [c_i, c_i + (2l-1)(tq-2q) - 2q + 1 - i(2(l-1)(tq-2q)+2)] \text{ and} \\
I_i &= [c_{i-1} + (2l-1)(tq-2q) - 3q + 2 - (i-1)(2(l-1)(tq-2q)+2), c_{i-1} + q - 1] \\
&= [c_{i-1}]_q \cap [c_{i-1} + (2l-1)(tq-2q) - 2q + 1 - i(2(l-1)(tq-2q)+2) + 1]_q
\end{aligned}$$

and the constants c_i, for $0 \leq i \leq m$, are chosen sufficiently far apart from each other, and from the $c_{i,j}^{[u,v]}$'s. These lists have size $|L(x_i)| = 2((l-1)(tq-2q)+1) + ((2l-1)(tq-2q) - 2q + 2 - i(2(l-1)(tq-2q)+2)) + (q - (2l-1)(tq-2q) + 3q - 2 + i(2(l-1)(tq-2q)+2) - (2(l-1)(tq-2q)+2)) = tq$.

Let us try to (p,q)-L-colour this graph. We may assume without loss of generality that u is precoloured a and v is precoloured b. Let us examine the subgraph $H_{a,b}$. We can prove inductively, by applying Lemma 1(ii), that the set of extendable colours for u_i^j is a subset of the interval $J_{i,j}^u$ for all $(i,j) \in \{0,\ldots,m\} \times \{1,\ldots,l-1\}$. Similarly, the set of extendable colours for v_i^j is a subset of $J_{i,j}^v$. It follows that the set of extendable colours for x_i is contained in $I_i \cup J_i$. Again, we inductively apply Lemma 1(ii) to show that the set of extendable colours for x_i is a subset of the interval J_i, for all $i \in \{1,2,\ldots,m\}$. However, $J_m = \emptyset$ and hence the graph is not (p,q)-L-colourable.

For the case $k = 2l+2$, it is routine to verify that an analogous list assignment, to account for the subdivisions, can be constructed in a similar inductive fashion. □

5.3 Upper bounds for graphs of bounded density

Here we shall study the link between the circular choice number of a graph and its density. Recall that the *maximum average degree* $\mathrm{Mad}(G)$ of a graph G is defined as the maximum over all subgraphs of G of the average degree. The aim of this subsection is to prove the following proposition:

Proposition 28 *The following hold:*

(i) *If* $\mathrm{Mad}(G) < k+1$ *then* $\mathrm{cch}(G) \leq 2k$.

(ii) *If* $\mathrm{Mad}(G) < 2 + \frac{2}{3n-1}$ *then* $\mathrm{cch}(G) \leq 2 + \frac{2}{n}$.

(iii) *If* $k \geq 2$ *and* $\mathrm{Mad}(G) < k+1+\frac{k+1-r}{2k+3-r}$ *with* $0 \leq r \leq k$ *then* $\mathrm{cch}(G) \leq 2k + \frac{2}{r+2}$.

(iv) *If* $k \geq 2$ *and* $\mathrm{Mad}(G) < k+1+\frac{k+1}{k+1+s}$ *with* $s \in \{1,2\}$ *then* $\mathrm{cch}(G) \leq 2k + \frac{4}{s+2}$.

Part (i) follows from the fact that if $\mathrm{Mad}(G) < k+1$, then $\delta^*(G) \leq k$ and we apply Lemma 7. The other two parts will follow from structural properties of graphs that are "critical" with respect to circular choice number. We define a graph G to be t-*critical* if $\mathrm{cch}(G) > t$ and $\mathrm{cch}(H) \leq t$ for every proper subgraph H of G.

Lemma 29 *Let k be a positive integer k and G be a $(2k+\alpha)$-critical graph with $\alpha \geq 0$. Then G has minimum degree at least $k+1$.*

Proof. As $\mathrm{cch}(G) > 2k+\alpha$, there exist $\varepsilon > 0$, two integers p, q and a $(2k+\alpha+\varepsilon)$-list-assignment L such that G cannot be (p,q)-L-coloured. Observe that every proper subgraph H of G can be (p,q)-L-coloured since, by the definition of G, $\mathrm{cch}(H) \leq 2k+\alpha$. Let $t = 2k+\alpha+\varepsilon$.

Suppose v is a vertex of degree at most k. By minimality of G, one can (p,q)-L-colour $G-v$. To apply Lemma 1(i), we can consider the colouring of $G-v$ as a precolouring and, hence, the number of extendable colours for v is at least $tq - k(2q-1) = \alpha q + \varepsilon q + k \geq 1$. This yields a (p,q)-L-colouring of G, a contradiction. □

For part (ii) of Proposition 28, we use the following lemma which is a direct consequence of Lemma 3.

Lemma 30 *Fix a positive integer n and let G be a $\left(2+\frac{2}{n}\right)$-critical graph. The graph G has minimum degree at least two and no handle of order more than $n-1$.*

Proof. Clearly, there can be no vertices with degree one or less. If there is a handle of order at least n, then any precolouring of the graph less the handle can be extended, by Lemma 3, to the entire graph, contradicting $\left(2+\frac{2}{n}\right)$-criticality. □

Proof of part (ii) **of Proposition 28.** We show that all graphs with circular choice number more than $2+\frac{2}{n}$ have maximum average degree at least $2+\frac{2}{3n-1}$. It suffices to prove this for $\left(2+\frac{2}{n}\right)$-critical graphs. Let G be such a graph. By Lemmas 29 and 30, G has minimum degree at least two and there are no handles of order more than $n-1$. We will use the following discharging procedure. At the beginning of the procedure, as usual, the charge of each vertex is equal to its degree. Then every vertex u of degree three or more gives charge $\eta = \frac{1}{3n-1}$ to each degree two vertex along the handles emanating from u. After the procedure, each degree two vertex has charge exactly $2+2\eta = 2+\frac{2}{3n-1}$. Since the order of every handle of G is at most $n-1$, each vertex with degree at least three will have new charge at least $3 - 3(n-1)\eta = 2 + \frac{2}{3n-1}$. Therefore, the average degree of G is at least $2 + \frac{2}{3n-1}$, as desired. □

For parts (iii) and (iv) of Proposition 28, we need the following lemma.

Lemma 31 *Let $k \geq 2$ be an integer, and G a $(2k+\alpha)$-critical graph with $\alpha \geq 0$. Then,*

(i) *the neighbours of a vertex of degree $k+1$ having degree $k+1$ are pairwise non-adjacent;*

(ii) *if $\alpha \geq \frac{2}{r+2}$, then a vertex of degree $k+1$ is adjacent to at most r vertices of degree $k+1$; and*

(iii) *for $s \in \{1,2\}$, if $\alpha \geq \frac{4}{s+2}$, then a vertex of degree $k+2$ is adjacent to at most s vertices of degree $k+1$.*

Proof. As $\mathrm{cch}(G) > 2k+\alpha$, there exist $\varepsilon > 0$, two integers p, q and a $(2k+\alpha+\varepsilon)$-list-assignment L such that G cannot be (p,q)-L-coloured. Let $t = 2k+\alpha+\varepsilon$.

(i) Let v be a vertex of degree $k+1$ in G. Let H be the graph induced by v and its neighbours of degree $k+1$ in G. If there is an edge between two of these neighbours of v, then H is not a star.

By minimality, there exists a (p,q)-L-colouring c of $G-H$. Let us now consider the list of extendable colours $L'(u)$ for a vertex u of H. Formally, $L'(u) = L(u) \setminus \bigcup_{w \in N(u) \setminus V(H)} [c(w)]_q$. The vertex u has a list $L(u)$ of size more than $(2k+\alpha)q$ and at most $2q-1$ colours are forbidden by each of its $k+1-d_H(u)$ previously coloured neighbours. Hence $|L'(u)| \geq (2k+\alpha)q - (2q-1)(k+1-d_H(u)) \geq (2d_H(u)-2)q$.

If H is not K_3 then, by Lemma 23, it admits a (p,q)-L'-colouring c'. By the definition of L', the union of c and c' is a (p,q)-L-colouring of G.

Suppose now that H is the complete graph on three vertices, and let x, y, z be the vertices of H. Since $k \geq 2$, there is a vertex u not in H adjacent to z. Observe that, if we consider the subgraph H' induced by $\{u, x, y, z\}$, then our goal is to (p,q)-L'-colour it, with $L'(u) = \{c(u)\}$, and $L'(a) = L(a) \setminus \bigcup_{w \in N(a) \setminus V(H')} [c(w)]_q$ for $a \in \{x, y, z\}$. Note that, for $a \in \{x, y, z\}$, the list $L'(a)$ has size at least $4q$ if au is an edge, and at least $2q$ otherwise — in particular, $|L'(z)| \geq 4q$. Hence, without loss of generality, we can assume that $L'(x)$ has size $2q$, and if xu is an edge, then these $2q$ colours are not in $[c(u)]_q$. The same holds for y. Therefore, the fact that c can be extended to G directly follows from part (ii) of Proposition 17.

(ii) Let v a vertex of degree $k+1$ in G. Suppose that v has $r+1$ neighbours $v_1, \dots, v_{r+1}$ of degree $k+1$. By (i), the vertices $v_1, \dots, v_{i+1}$ are pairwise non-adjacent; this observation is important for the valid application of Lemma 1. Consider any (p,q)-L-colouring of $G - \{v_1, \dots, v_{r+1}, v\}$. Using Lemma 1, the number of extendable colours for v_i, for any $i \in \{1, 2, \dots, r+1\}$, is at least $tq - k(2q-1) = \alpha q + k + \varepsilon q$. Applying

Lemma 1(i) once again, the number of extendable colours for v is at least $tq-(k-r)(2q-1)-(r+1)(2q-(\alpha q+k+\varepsilon q)) = 2(\alpha-1)q+2(k+\varepsilon q)+r(\alpha q-1+k+\varepsilon q) = (\alpha(r+2)-2)q+2(k+\varepsilon q)+r(k+\varepsilon q-1) \geq \left(\frac{2}{r+2}(r+2)-2\right)q+1 \geq 1$, since $r \geq \min\left\{0, \frac{s-2}{2}\right\}$ and $k \geq 1$, a contradiction to t-criticality.

(iii) Suppose v is a vertex of degree $k+2$ with $s+1$ neighbours $v_1, \ldots, v_{s+1}$ of degree $k+1$. Consider any (p,q)-L-colouring of $G-\{v_1,\ldots,v_{s+1},v\}$. By (ii), the vertices $v_1,\ldots,v_{s+1}$ are mutually non-adjacent, since $\alpha \geq 1$. Hence, using Lemma 1(i), the number of extendable colours for v_i, for any $i \in \{1,2,\ldots,s+1\}$, is at least $tq-k(2q-1) = \alpha q+k+\varepsilon q$. Applying the lemma again, the number of extendable colours for v is at least $tq-(k+1-s)(2q-1)-(s+1)(2q-(\alpha q+k+\varepsilon q)) = 2(\alpha-2)q+2(k+\varepsilon q)+1+s(\alpha q-1+k+\varepsilon q) = (\alpha(s+2)-4)q+2(k+\varepsilon q)+s(k+\varepsilon q-1)+1 \geq 1$, since $\alpha \geq \frac{4}{s+2}$ and $k \geq 1$, a contradiction to t-criticality.

□

Proof of part (iii) **of Proposition 28**. Suppose that G is $\left(2k+\frac{2}{r+2}\right)$-critical. Then, by Lemmas 29 and 31, G has minimum degree at least $k+1$, and a vertex of degree $k+1$ is adjacent to at most r vertices of degree $k+1$. We use the following vertex discharging procedure: first, every vertex has charge equal to its degree; then, every vertex of degree at least $k+2$ gives charge $\eta = \frac{1}{2k+3-r}$ to each of its neighbours of degree $k+1$.

We want to show now that the new charge of every vertex v is at least $k+1+(k+1-r)\eta$. If v has degree $k+1$, then its new charge is at least $k+1+(k+1-r)\eta$ since all of its neighbours of degree more than $k+1$ gave it charge. If v has degree at least $k+2$, then its new charge is at least $k+2-(k+2)\eta = k+1+(k+1-r)\eta$.

It follows now that the average degree, and hence the maximum average degree, is at least $k+1+(k+1-r)\eta$. This shows that every graph with circular choice number more than $2k+\frac{2}{r+2}$ has maximum average degree at least $k+1+(k+1-r)\eta$, as required.

□

Proof of part (iv) **of Proposition 28**. Suppose that G is $\left(2k+\frac{4}{s+2}\right)$-critical. Then, by Lemmas 29 and 31, G has minimum degree at least $k+1$, no two vertices of degree $k+1$ are adjacent, and any vertex of degree $k+2$ is adjacent to at most s vertices of degree $k+1$. We use the following vertex discharging procedure: first, every vertex has charge equal to its degree; then, every vertex of degree at least $k+2$ gives charge $\eta = \frac{1}{k+1+s}$ to each of its neighbours of degree $k+1$.

We want to show now that the new charge of every vertex v is at least $k+1+(k+1)\eta$. If v has degree $k+1$, then its new charge is at least $k+1+(k+1)\eta$ since all of its neighbours gave it charge. If v has degree $k+2$, then its new charge is at least $k+2-s\eta = k+1+(k+1)\eta$ since it has at most s neighbours of degree $k+1$ to give charge to. If v has degree at least $k+3$, then its new charge is at least $k+3-(k+3)\eta = k+1+(k+2s-1)\eta \geq k+1+(k+1)\eta$ since $s \geq 1$.

It follows now that the average degree, and hence the maximum average degree, is at least $k+1+(k+1)\eta$. This shows that every graph with circular choice number more than $2k+\frac{4}{s+2}$ has maximum average degree at least $k+1+(k+1)\eta$, as required. $\square$

5.4 Upper bounds for graphs of bounded density and prescribed girth

Our eventual goal is to apply results of the previous subsection to planar graphs of prescribed girth in Subsection 5.5 (since such graphs have bounded maximum average degree); however, we can get improvements by taking advantage of the girth requirement. In this subsection, we will prove the following three assertions. The first one is a strengthening of part (ii) of Proposition 28 when the girth is at least $2n+2$, and the last two are strengthenings of parts (iii) and (iv) of Proposition 28 when the graph under consideration has girth at least 4 and 6, respectively.

Proposition 32 *Let k, s be positive integers and set $r = \left\lceil \frac{s-2}{2} \right\rceil$.*

(i) *If G has girth at least $2n+2$ and $\mathrm{Mad}(G) \le 2+\frac{1}{n}$, then $\mathrm{cch}(G) \le 2+\frac{2}{n}$.*

(ii) *If $s \le 2k+2$, G has girth at least four and $\mathrm{Mad}(G) < k+1+\frac{k+1-r}{k+1+s'-r}$, where $s' = \min\{k+2, s\}$, then $\mathrm{cch}(G) \le 2k+\frac{4}{s+2}$.*

(iii) *If $2 \le s \le k+2$, G has girth at least 6 and $\mathrm{Mad}(G) < k+1+\frac{k+1-r}{k+s-r+\frac{1}{k+3-s}}$, then $\mathrm{cch}(G) \le 2k+\frac{4}{s+2}$.*

Before giving the lemma that will be used to prove part (i) of Proposition 32, let us introduce some notation. Let G be a graph. We define a *chain* to be a maximal handle in G. Its *order* is the number of its vertices. (A chain of order 0 is just an edge between vertices of degree at least three.) We say that two vertices of degree at least three in G are *linked with weight l* if there is a chain of order l between them.

Lemma 33 *Fix a positive integer n and let G be a graph that has girth at least $2n+2$ and is $\left(2+\frac{2}{n}\right)$-critical. Let v be a vertex of degree $d \ge 3$. Denote by $w_1, w_2, \ldots, w_d$ the weights of the chains emanating from v. Then, $\sum_{j=1}^{d} w_j \le (d-1)n-2$.*

Proof. Similarly to previous lemmas, we begin with a $(t+\varepsilon)$-(p,q)-list-assignment L, where $t = 2+\frac{2}{n}$ and $\varepsilon > 0$, such that G is not (p,q)-L-colourable. Notice that, according to Lemma 30, there are no handles of order at least n in G. Then we take a (p,q)-L-precolouring of the subgraph G less the vertex v, the chains incident to it. We moreover assume that $\sum_{j=1}^{d} w_j \ge (d-1)n-1$. Applying Lemma 1(i) repetitively for each chain — which is made possible by the girth requirement — shows that, if u_j is a neighbour of v belonging to a chain of order w_j, then the number of extendable colours for u_j is at least $w_j\frac{2q}{n}+1$. Therefore, by applying once again the lemma, we infer that the number of extendable colours for v is at least

$$2q + \frac{2q}{n} + \varepsilon q - \sum_{j=1}^{d} \left(2q - \left(w_j \frac{2q}{n} + 1\right)\right) \geq \frac{4q}{n} + \varepsilon q + d \geq 1,$$

since $\sum_{j=1}^{d} w_j \geq (d-1)n - 1$, a contradiction. □

Proof of part (i) **of Proposition 32**. As usual, we perform a discharging procedure to show that all graphs with circular choice number more than $2 + \frac{2}{n}$ and girth at least $2n+2$ have maximum average degree at least $2 + \frac{1}{n}$. It suffices to prove this for $\left(2 + \frac{2}{n}\right)$-critical graphs (of girth at least $2n+2$). Let G be such a graph. By Lemma 30, G has minimum degree at least two and there are no handles containing more than $n-1$ vertices. We use the following discharging procedure. At the beginning of the procedure, as usual, the charge of each vertex is equal to its degree. Then, every vertex of degree at least three gives charge $\eta = \frac{1}{2n}$ to each degree two vertex along the handles emanating from it. At the end, each degree two vertex has charge exactly $2 + 2\eta = 2 + \frac{1}{n}$. Furthermore, by Lemma 33, the new charge of every vertex of degree $d \geq 3$ is at least

$$d - ((d-1)n - 2) \cdot \frac{1}{2n} = \frac{d+1}{2} + \frac{1}{n} \geq 2 + \frac{1}{n},$$

which concludes the proof. □

To prove parts (ii) and (iii) of Proposition 32, we require the following lemma.

Lemma 34 *Let G be a $(2k+\alpha)$-critical graph of girth g and $s \in \{0, 1, \dots, k+1\}$.*

(i) *If $g \geq 4$ and $\alpha \geq \frac{4}{s+2}$, then a vertex of degree $k+2$ is adjacent to at most s vertices of degree $k+1$.*

We define a hibernian *to be a vertex of degree $k+2$ with s neighbours of degree $k+1$. A* barbarian *is a vertex of degree $k+2$ with exactly $s-1$ neighbours of degree $k+1$.*

(ii) *If $g \geq 5$ and $\alpha \geq \frac{3}{s+1}$, then two hibernians cannot be adjacent.*

(iii) *If $g \geq 6$ and $\alpha \geq \frac{4}{3s+1}$, then a barbarian is adjacent to at most one hibernian.*

Proof. Take ε, p, q, L and t as in the proof of Lemma 31.

(i) The proof is largely similar to the proof of Lemma 31(iii), so we omit it here. Note that the condition $g \geq 3$ ensures that the neighbours of a vertex are pairwise non-adjacent.

(ii) Suppose v and v' are adjacent hibernians. Denote the s neighbours of v of degree $k+1$ by $V_v = \{v_1, \dots, v_s\}$, and those of v' by $V_{v'} = \{v'_1, \dots, v'_s\}$. Since the girth of G is at least five, V_v and $V_{v'}$ are disjoint; furthermore, $V_v \cup V_{v'}$ induces an independent set in G. This observation will make our subsequent applications of Lemma 1(i) valid.

Consider any (p,q)-L-colouring of $G \setminus (V_v \cup V_{v'} \cup \{v, v'\}$. We wish to show that such a colouring can be extended to the entire graph G, giving a contradiction to t-criticality. Let us use the vertex ordering as just given; it is clear that this ordering satisfies the properties for Lemma 1(i).

First, the number of extendable colours for v_i or v_i', for any i, is at least

$$x_0 = tq - k(2q-1) = \alpha q + k + \varepsilon q.$$

We have $x_0 \geq 1$, since $k \geq 1$ and $\alpha, \varepsilon \geq 0$. Now, since v is adjacent to $v_1, \ldots, v_s$ and $k+1-s$ precoloured vertices, the number of extendable colours for v is at least

$$\begin{aligned} x &= tq - s(2q - x_0) - (k+1-s)(2q-1) \\ &= ((s+1)\alpha - 2)q + k + \varepsilon q + s(k + \varepsilon q - 1) + 1. \end{aligned}$$

We have $x \geq 1$, since $\alpha s \geq 2$ and $k \geq 1$. Applying the lemma again, since v' is adjacent to $v_1', \ldots, v_s'$, $k+1-s$ precoloured vertices and v, it follows that the number of extendable colours for v' is at least

$$\begin{aligned} x' &= tq - s(2q - x_0) - (k+1-s)(2q-1) - (2q - x) \\ &= ((s+1)\alpha - 3)2q + 2\{k + \varepsilon q + s(k + \varepsilon q - 1) + 1\}. \end{aligned}$$

Since $\alpha \geq \frac{3}{s+1}$ then $x' \geq 0$. But this means that G is (p,q)-L-colourable, a contradiction.

(iii) Suppose a barbarian w is adjacent to two hibernians v and v'. Since the girth is at least six the neighbourhoods of w, v and v' are pairwise disjoint and their union is an independent set. Consider any (p,q)-L-colouring of $G \setminus (\{v_1, \ldots, v_s, v_1', \ldots, v_s', w_1, \ldots, w_{s-1}, v, v', w\}$ and extend the colouring using the ordering above via Lemma 1(i). As in (ii) every neighbour of v, v' or w has at least $x_0 = \alpha q + k + \varepsilon q$ extendable colours, and v and v' have at least

$$x = ((s+1)\alpha - 2)q + k + \varepsilon q + s(k + \varepsilon q - 1) + 1$$

extendable colours.

Now the number of extendable colours at w is at least

$$tq - (s-1)(2q - x_0) - (k+1-s)(2q-1) - 2(2q - x) \geq ((3s+1)\alpha - 4)q + 1.$$

Since $\alpha \geq \frac{4}{3s+1}$ then G is (p,q)-L-colourable, a contradiction. □

The proof of part (ii) of Proposition 32 follows from the part (i) of the previous lemma and is very similar to that of part (iv) of Proposition 28, so we omit it here.

Proof of part (iii) **of Proposition 32**. Suppose that G is $\left(2k + \frac{4}{s+2}\right)$-critical and has girth at least 6. Then, by Lemmas 29 and 34, G has minimum degree at least $k+1$, any vertex of degree $k+1$ is adjacent to at most r vertices of degree $k+1$, and any vertex of degree $k+2$ is adjacent to at most s vertices of degree $k+1$; furthermore, as $s \geq 2$, $\frac{4}{s+2} \geq \frac{4}{3s+1}$, no two hibernians are adjacent and every barbarian is adjacent to at most one hibernian. We shall use the following vertex discharging procedure: first, every has charge equal to its degree; then, every vertex of degree $k+1$ receives charge $\eta = \frac{1}{k+s-r+\frac{1}{k+3-s}}$ from every neighbour of degree at least $k+2$, and every hibernian receives charge $\eta_1 = \frac{\eta}{k+3-s} = \frac{1}{(k+s-r)(k+3-s)+1}$ from every non-hibernian neighbour of degree at least $k+2$.

We want to show now that the new charge of every vertex is at least $k+1+(k+1-r)\eta$. If v has degree $k+1$, then its new charge is at least $k+1+(k+1-r)\eta$ since it has at least $k+1-r$ neighbours that gave it charge η. If v is a hibernian, then its new charge is at least $k+2-s\eta+(k+2-s)\eta_1 = k+1+\left(\frac{1}{\eta}-s+\frac{k+2-s}{k+3-s}\right)\eta = k+1+(k+1-r)\eta$ since it has at most s neighbours of degree $k+1$ to give charge to and it receives η_1 charge from all of the other adjacent vertices. If v is a barbarian, then its new charge is at least $k+2-(s-1)\eta-\eta_1 = k+1+\left(\frac{1}{\eta}-s+1-\frac{1}{k+3-s}\right)\eta = k+1+(k+1-r)\eta$ since it has $s-1$ neighbours of degree $k+1$ and at most one hibernian neighbour. If v is a non-hibernian, non-barbarian vertex of degree $k+2$, then its new charge is at least $k+2-(s-2)\eta-(k+2-(s-2))\eta_1 = k+1+\left(\frac{1}{\eta}-s+2-\frac{k+4-s}{k+3-s}\right)\eta = k+1+(k+1-r)\eta$ since it has at most $s-2$ neighbours of degree $k+1$ (and the rest could be hibernians). If v is a vertex of degree at least $k+3$, then its new charge is at least $k+3-(k+3)\eta \geq k+1+(k+1-r)\eta$; this holds if and only if $k+s-r+\frac{1}{k+3-s} \geq k+2-\frac{r}{2} \iff s-\frac{r}{2}-2+\frac{1}{k+3-s} \geq 0$ and this is true when $s \geq 2$.

It follows now that the average degree, and hence the maximum average degree, is at least $k+1+(k+1-r)\eta$. This shows that every graph with circular choice number more than $2k+\frac{4}{s+2}$ and girth at least 6 has maximum average degree at least $k+1+(k+1-r)\eta$, as required. □

5.5 Upper bounds for planar and toroidal graphs of prescribed girth

With Propositions 28 and 32 in hand, it is straightforward to give upper bounds for $\tau(k)$.

Theorem 35 *The following holds:*

(i) $\tau(4) \leq 6$; $\tau(5) \leq 4+\frac{1}{2}$; $\tau(6) \leq 4$; $\tau(8) \leq 3+\frac{1}{3}$; $\tau(9) \leq 3$; *and*

(ii) $\tau(4l+2) \leq 2+\frac{2}{l}$ *for any* $l \geq 1$.

Proof. By Euler's formula, every planar graph with girth at least k has maximum average degree less than $\mathrm{Mad}_k = 2+\frac{4}{k-2}$.

(i) $\mathrm{Mad}_4 = 4$ so, by part (i) of Proposition 28, $\tau(4) \leq 6$. $\mathrm{Mad}_5 = 3+\frac{1}{3}$ and, setting $k=2$ and $s=6$, part (ii) of Proposition 32 gives $\tau(5) \leq 4+\frac{1}{2}$. $\mathrm{Mad}_6 = 3$ and, by part (i) of Proposition 28, $\tau(6) \leq 4$. $\mathrm{Mad}_8 = 2 + \frac{2}{3}$ and, setting $k = 1$ and $s = 1$, part (iv) of Proposition 28 gives $\tau(8) \leq 3\frac{1}{3}$. $\mathrm{Mad}_9 = 2\frac{4}{7}$ and, setting $k = 1$ and $s = 2$, part (iii) of Proposition 32 gives $\tau(9) \leq 3$.

(ii) This is a direct consequence of part (i) of Proposition 32.

□

Results of the previous two subsections should also give analogous upper bounds for graphs with prescribed girth embeddable upon higher surfaces. For example, a graph is *toroidal* if it can be embedded without crossing edges on the torus. It follows from the Euler-Poincaré formula that the maximum average degree of a toroidal graph of girth at least k is *at most* $2 + \frac{4}{k-2}$. If we define

$$\tau_t(k) := \sup\{\mathrm{cch}(G) : G \text{ is toroidal and has girth} \geq k\},$$

then it follows from similar calculations to those of Theorem 35 that $\tau_t(3) \leq 11, \tau_t(4) \leq 6\frac{2}{5}, \tau_t(5) \leq 4 + \frac{4}{5}, \tau_t(6) \leq 4\frac{1}{2}, \tau_t(7) \leq 4, \tau_t(9) \leq 3\frac{1}{3}, \tau_t(10) \leq 3$, and $\tau_t(6l+1) \leq 2 + \frac{2}{l}$ for any $l \geq 1$. More generally, if a surface $\mathcal{S}$ has Euler characteristic ε, then, using the Euler-Poincaré formula, it follows that a graph G embeddable upon $\mathcal{S}$ with girth at least k has maximum average degree at most $2 + \frac{2}{k-2} - \frac{2\varepsilon}{|V(G)|}$. We then apply one of the parts of Propositions 28 or 32, as appropriately. In particular, the upper bounds for t_t will indeed also hold for graphs embeddable on any surface with nonnegative Euler characteristic.

As for graphs of genus $r \geq 2$, let us define $f(\mathrm{Ad}, k) = 2\sum_{i=0}^{s-1}(\mathrm{Ad}-1)^i$ if $k = 2s$ and $f(\mathrm{Ad}, k) = 1 + \mathrm{Ad}\sum_{i=0}^{s-1}(\mathrm{Ad}-1)^i$ if $k = 2s+1$. It has been proved in [2] that any graph H of girth k satisfies $|V(H)| \geq f(\mathrm{Ad}, k)$. Hence, we deduce that the maximum average degree of any graph G of genus r and girth k satisfies

$$f(\mathrm{Mad}(G), k)\left(\mathrm{Mad}(G) - \frac{2k}{k-2}\right) - \frac{4(r-1)k}{k-2} \leq 0. \tag{3}$$

Thus, we can easily deduce conditions in terms of girth and genus that allow to use the theorems of the previous two subsections. For instance, we obtain corollaries of the following type.

Corollary 36 *Every triangle-free graph of genus three has circular choice number at most* $8 + \frac{2}{3}$.

Proof. We infer from (3) that the maximum average degree of such a graph is at most $2+2\sqrt{3}$. Therefore, applying part (ii) of Proposition 32 with $k = s = 4$ yields the conclusion. □

We do not yet see how to get better examples for analogous lower bounds for higher surfaces.

5.6 Outerplanar graphs of prescribed girth

An outerplanar graph is a graph that can be drawn in the plane such that the outer face contains every vertex of the graph. For any $k \geq 3$, we define

$$\tau_o(k) = \sup\{\mathrm{cch}(G) : G \text{ is outerplanar and has girth} \geq k\}$$

We will show the following theorem:

Theorem 37 *$\tau_o(k) = 2 + \frac{2}{k-2}$ for all integers $k \geq 3$.*

We start by exhibiting a class of examples that show that $\tau_o(k)$ is at least the expression given in the theorem.

Proposition 38 *Fix $k \geq 3$. For any integer $n \geq 1$, there exists an outerplanar graph $O_{k,n}$ of girth k whose circular choice number is at least $2 + \frac{2}{k-2} - \frac{1}{n}$.*

Proof outline. Let $t = 2 + \frac{2}{k-2} - \frac{1}{n}$; choose $q = 2(k-2)n$; and choose p much larger than tq. We consider the graph P_m of Figure 4, with m large enough. We construct the

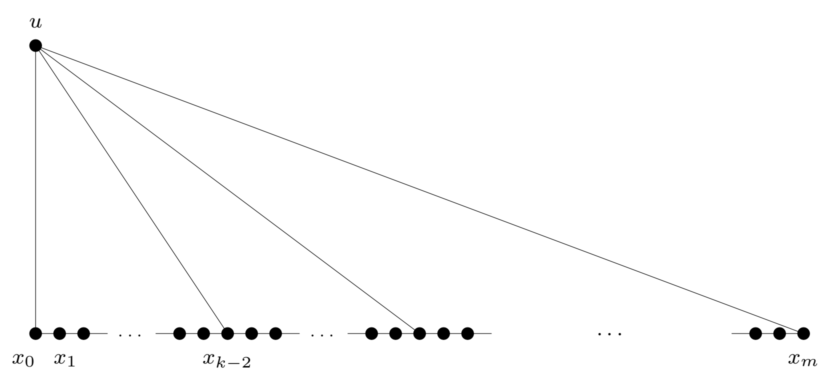

Figure 4: The graph P_m.

graph $O_{k,n}$ by taking tq copies of P_m and identifying the vertices u of each copy. For each $a \in L(u)$, we can associate a copy P_a of P_m and a corresponding list assignment such that, if u is coloured a, then the subgraph P_a cannot be (p,q)-L-coloured. As the calculations are largely similar to those of Propositions 26 and 27, and a bit tedious, we just give the precise definition of the lists for vertices $x_0, x_1, \ldots, x_{k-2}$ of the subgraph P_a, and give an argument to show that, provided that m is large enough and u is coloured a, the subgraph P_a cannot be (p,q)-L-coloured. Set

$$J_t = [c_t, c_t + (4n - 2(k-2))(t+1)] \text{ for } i \in \{0, \ldots, k-3\}, \text{ and}$$

$I_t = [c_{t-1} + 4nt - 2(n+t)(k-2) + 1, c_{t-1} + 2(k-2)n - 1]$ for $i \in \{1, \ldots, k-2\}$.

Let

$$L(x_i) = I_i \cup J_i \text{ for } i \in \{1, \ldots, k-3\},$$
$$L(x_0) = [a]_q \cup J_0, \text{ and}$$
$$L(x_{k-2}) = [a_q] \cup I_{k-2} \cup J_0',$$

where $J_0' = [c_0, c_0 + 4n - 2(k-2) - 2(k-2)^2 + 1]$. Each list has size tq. Suppose now that u is coloured a, and let us try to define a colouring c of the vertices $x_0, \ldots, x_{k-2}$. Clearly, $c(x_0) \in J_0$ and thus, by applying Lemma 1, $c(x_i) \in J_i$ for every $i \in \{0, \ldots, x_{k-3}\}$. Therefore, $c(x_{k-2}) \in J_0'$. Notice that $|J_0'| < |J_0|$ (since $k \geq 3$). Hence, by defining analogously the lists for $x_{k-1}, \ldots, x_m$ we get that x_m cannot be coloured, provided that m is large enough. □

Next we will show that $\tau_o(k)$ is also at most the expression in the theorem. We note that the bound appears in [13] for even k. We give a more succinct presentation of the proof and also cover the case for odd k. The following lemma follows from Lemma 3.

Lemma 39 *Fix $k \geq 3$. Let L be a $\left(2 + \frac{2}{k-2}\right)$-$(p,q)$-list-assignment of the cycle C_k. Every precolouring of two adjacent vertices on the cycle can be extended to a (p,q)-L-colouring of the entire cycle.*

Proposition 40 *Every outerplanar graph of girth at least $k \geq 3$ has circular choice number at most $2 + \frac{2}{k-2}$.*

Proof. Suppose there exists an outerplanar graph G of girth at least k that is $\left(2 + \frac{2}{k-2}\right)$-critical. Let L be a $\left(2 + \frac{2}{k-2} + \varepsilon\right)$-$(p,q)$-list-assignment of G such that G is not (p,q)-L-colourable. Clearly, G cannot have any leaves (vertices of degree one). We now note that every outerplanar graph of girth at least k with no leaves can be inductively built up from a collection of cycles of length at least k as follows. Initially, we start with a cycle of length at least k; at each step of the construction, we identify an edge of a new cycle with an edge of the outerplanar graph constructed till that point. It is clear that we can also inductively apply Lemma 39 to produce a (p,q)-L-colouring of G, a contradiction. □

6 Concluding remarks

We have made efforts to tackle some problems in this new line of research. We showed that the difference between the circular choice number and circular chromatic number of a circular clique is unbounded. We provided some evidence in support of a positive answer to the question of Problem 8, by showing that $\mathrm{cch}(G) = O(\mathrm{ch}(G) + \ln|V(G)|)$. We showed, perhaps counterintuitively, that the value of τ lies between 6 and 8, rather than between 4 and 5 — as was suggested by Mohar [8].

Much further work, however, remains. Problem 8 remains open as do other fundamental questions posed by Zhu:

Problem 41 *Is the circular choice number of every graph a rational? Is it always attained?*

Also, with regard to toroidal and planar graphs (of prescribed girth), most of the exact values of $\tau_t(k)$ and $\tau(k)$ are unknown. In Table 1, we summarise the bounds on these parameters that we have obtained.

k (girth)	3	4	5	6	7	8	9	10	$k \geq 11$
$\tau_t(k)$ upper	11	$6+\frac{2}{5}$	$4+\frac{4}{5}$	$4+\frac{1}{2}$	4	4	$3+\frac{1}{3}$	3	$2+\frac{4}{2\lfloor (k-3)/4 \rfloor}$
$\tau(k)$ upper	8	6	$4+\frac{1}{2}$	4	4	$3+\frac{1}{3}$	3	3	$2+\frac{4}{2\lfloor (k-2)/4 \rfloor}$
$\tau(k)$ lower	6	4	$3+\frac{1}{3}$	3	$2+\frac{4}{5}$	$2+\frac{2}{3}$	$2+\frac{4}{7}$	$2+\frac{1}{2}$	$2+\frac{4}{k-2}$

Table 1: Bounds for toroidal and planar graphs of prescribed girth k.

It is unclear how the examples for the lower bounds can be generalised to give higher circular list chromatic numbers for graphs on higher surfaces.

It should also be mentioned, as has been hinted at several times in the paper, that circular choosability is closely related to the notion of T-choosability, which was introduced by Tesman [10] and further studied by, for instance, Alon and Zaks [3]. Given a set T of forbidden differences, a T-proper colouring of G is a colouring that satisfies $|c(v)-c(u)| \notin T$ whenever $uv \in E(G)$. In T-choosability T-ch(G) of G is the least number k such that any list assignment L with $|L(v)| \geq k$ for all $v \in V(G)$ admits a T-proper colouring c with $c(v) \in L(v)$ for all v. An important special case that has received a lot of special attention is the case when $T = T_r = \{0, \ldots, r\}$. By considering the proofs it should be clear that many of the bounds on $\mathrm{cch}(G)$ given in this paper extend to T_r-ch(G) when multiplied by $r+1$. However, as might be expected, the "circularity constraint" is an essential difference between T_r-choosability and circular choosability. It is for instance known that $T_r\text{-ch}(C_{2n}) = \left\lfloor \frac{4n-2}{4n-1}(2r+2) \right\rfloor + 1$, see [9], whereas we know that $\mathrm{cch}(C_{2n}) \geq \mathrm{cch}(K_2) = 2$. It has recently come to our attention that the T_r-analogue of Problem 8 has been investigated by Waters [14], who has shown that T_r-ch$(G) \leq 2(r+1)$ for all graphs with $\mathrm{ch}(G) \leq 2$.

References

[1] N. Alon. Degrees and choice numbers. *Random Structures Algorithms*, 16(4):364–368, 2000.

[2] N. Alon, S. Hoory, and N. Linial. The Moore bound for irregular graphs. *Graphs Combin.*, 18(1):53–57, 2002.

[3] N. Alon and A. Zaks. T-choosability in graphs. *Discrete Appl. Math.*, 82(1-3):1–13, 1998.

[4] O. V. Borodin. Criterion of chromaticity of a degree prescription (in Russian). In *Abstracts of IV All-Union Conf. on Theoretical Cybernetics (Novosibirsk)*, pages 127–128, 1977.

[5] P. Erdős, A. L. Rubin, and H. Taylor. Choosability in graphs. In *Proceedings of the West Coast Conference on Combinatorics, Graph Theory and Computing (Humboldt State Univ., Arcata, Calif., 1979)*, Congress. Numer., XXVI, pages 125–157, Winnipeg, Man., 1980. Utilitas Math.

[6] N. Gazit and M. Krivelevich. On the asymptotic value of the choice number of complete multi-partite graphs. *J. Graph Theory*, 52(2):123–134, 2006.

[7] S. Janson, T. Łuczak, and A. Rucinski. *Random graphs.* Wiley-Interscience Series in Discrete Mathematics and Optimization. Wiley-Interscience, New York, 2000.

[8] B. Mohar. Choosability for the circular chromatic number, 2003. http://www.fmf.uni-lj.si/mohar/Problems/P0201ChoosabilityCircular.html.

[9] R. Sitters. A short proof of a conjecture on the T_r-choice number of even cycles. *Discrete Appl. Math.*, 92:243–246, 1999.

[10] B.A. Tesman. List T-colorings of graphs. *Discrete Appl. Math.*, 45(3):277–289, 1993. ARIDAM IV and V (New Brunswick, NJ, 1988/1989).

[11] C. Thomassen. Every planar graph is 5-choosable. *J. Combin. Theory Ser. B*, 62(1):180–181, 1994.

[12] C. Thomassen. Color-critical graphs on a fixed surface. *J. Combin. Theory Ser. B*, 70(1):67–100, 1997.

[13] G. Wang, G. Liu, and J. Yu. Circular list colorings of some graphs. *J. Appl. Math. Comput.*, 20(1-2):149–156, 2006.

[14] R. J. Waters. Some new bounds on T_r-choosability. Submitted.

[15] X. Zhu. Circular choosability of graphs. *J. Graph Theory*, 48(3):210–218, 2005.

Unité de recherche INRIA Sophia Antipolis
2004, route des Lucioles - BP 93 - 06902 Sophia Antipolis Cedex (France)

Unité de recherche INRIA Futurs : Parc Club Orsay Université - ZAC des Vignes
4, rue Jacques Monod - 91893 ORSAY Cedex (France)
Unité de recherche INRIA Lorraine : LORIA, Technopôle de Nancy-Brabois - Campus scientifique
615, rue du Jardin Botanique - BP 101 - 54602 Villers-lès-Nancy Cedex (France)
Unité de recherche INRIA Rennes : IRISA, Campus universitaire de Beaulieu - 35042 Rennes Cedex (France)
Unité de recherche INRIA Rhône-Alpes : 655, avenue de l'Europe - 38334 Montbonnot Saint-Ismier (France)
Unité de recherche INRIA Rocquencourt : Domaine de Voluceau - Rocquencourt - BP 105 - 78153 Le Chesnay Cedex (France)

Éditeur
INRIA - Domaine de Voluceau - Rocquencourt, BP 105 - 78153 Le Chesnay Cedex (France)
http://www.inria.fr
ISSN 0249-6399